112. On Paracompactness of Topological Spaces

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K. Morita [5] proved that a topological space having the weak topology with respect to a closed covering is paracompact and normal if and only if each element of the closed covering is paracompact and normal.

In this note we shall investigate the paracompactness of a topological space X by the open covering of X.

Theorem 1. *Let $\{G_\alpha \mid \alpha \in \Omega\}$ be a locally finite open covering of a topological space X. Then X is paracompact*) if and only if each $\bar{G}_\alpha$ is paracompact.*

Proof. Let $\mathfrak{U}=\{U_\beta \mid \beta \in \Lambda\}$ be an arbitrary open covering of X. Since each $\bar{G}_\alpha$ is paracompact, for each α we have a locally finite open covering $\{V_\gamma \frown \bar{G}_\alpha \mid \gamma \in \Gamma\}$ of $\bar{G}_\alpha$ which is a refinement of $\{U_\beta \frown \bar{G}_\alpha \mid \beta \in \Lambda\}$ and each V_γ is an open set of X. By the closedness of $\bar{G}_\alpha$, $\{V_\gamma \frown \bar{G}_\alpha \mid \gamma \in \Gamma\}$ is locally finite in X and, so $\{V_\gamma \frown G_\alpha \mid \gamma \in \Gamma\}$ is a locally finite collection of open sets (in X). By the hypothesis of the locally finiteness of $\{G_\alpha \mid \alpha \in \Omega\}$ $\{V_\gamma \frown G_\alpha \mid \gamma \in \Gamma, \alpha \in \Omega\}$ is a locally finite open covering of X which is a refinement of $\mathfrak{U}$. This completes the proof.

Theorem 2. *Let $\mathfrak{U}=\{G_\alpha \mid \alpha \in \Omega\}$ be a star-finite open covering of a regular T_1 space X. If each G_α is paracompact as a subspace of X and each $\mathrm{Fr}(G_\alpha)$ has the Lindelöf property, then X is paracompact.*

Theorem 3. *Let $\mathfrak{U}=\{G_\alpha \mid \alpha \in \Omega\}$ be a locally finite open covering of a regular T_1 space X. If each G_α is paracompact and each $\mathrm{Fr}(G_\alpha)$ is compact, then X is paracompact.*

Proceeding the proof of the above theorems, we prove the following lemmas.

Lemma 1. *Let $\{G_1, G_2\}$ be an open covering of a regular T_1 space X such that $G_i (i=1,2)$ are paracompact as subspaces of X. In order that X is paracompact it is necessary and sufficient that there are mutually disjoint open sets $H_i (i=1,2)$ of X such that $H_i \subset G_j (i \neq j; i,j=1,2)$ and $H_i \supset \mathrm{Fr}(G_i)$ $(i=1,2)$.*

Proof. Necessity. By [2] X is normal. Since $\mathrm{Fr}(G_i)$ $(i=1,2)$ are closed in X and are mutually disjoint, there exist open sets $H_i (i=1,2)$ satisfying the above conditions.

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*) Topological space X is said to be *paracompact* if each open covering of X has a locally finite open covering as a refinement.

Sufficiency. Let $H_i(i=1,2)$ be open sets satisfying the above conditions. To prove that X is paracompact, let $\mathfrak{U}=\{U_\alpha \mid \alpha\in\Omega\}$ be an arbitrary open covering of X. Since $\mathrm{Fr}(G_1)$ is closed in G_2 and so is paracompact, there is a locally finite open covering $\{V'_\beta \mid \beta\in\Lambda\}$ of $\mathrm{Fr}(G_1)$ which refines $\{U_\alpha\frown\mathrm{Fr}(G_1) \mid \alpha\in\Omega\}$. By [1] G_2 is collectionwise normal as a subspace of X, there is a collection of open sets $\{V_\beta \mid \beta\in\Lambda\}$ of X such that it is locally finite in G_2 and $\underset{\beta\in\Lambda}{\smile} V_\beta\supset\mathrm{Fr}(G_1)$ and for each β $V_\beta\subset H_1$, $V_\beta\subset$ some U_α of $\mathfrak{U}$ and $V_\beta\frown\mathrm{Fr}(G_1)\subset V'_\beta$ (see [3]). Since $H_1\frown H_2=\phi$, $\{V_\beta \mid \beta\in\Lambda\}$ is locally finite collection of open sets of X. Let $V=\underset{\beta\in\Omega}{\smile} V_\beta$, then it is open in X and $H_1\supset V\supset\mathrm{Fr}(G_1)$. Since G_2 is normal, there are open sets A, B of X such that $V\supset\bar{A}\supset A\supset\bar{B}\supset B\supset\mathrm{Fr}(G_1)$ where bars can show the closures in X by $V\frown\mathrm{Fr}(G_2)=\phi$.

By the closedness of G_1-A in G_1 it is paracompact, so there exists a locally finite open covering $\{W_\gamma^{(1)\prime} \mid \gamma\in\Gamma\}$ of G_1-A which refines $\{U_\alpha\frown(G_1-A) \mid \alpha\in\Omega\}$. By the collectionwise normality of G_1 there is a locally finite collection $\{W_\gamma^{(1)} \mid \gamma\in\Gamma\}$ of open sets of X such that for each $\gamma\in\Gamma$ $W_\gamma^{(1)}\frown(G_1-A)\subset W_\gamma^{(1)\prime}$, $W_\gamma^{(1)}\subset G_1-\bar{B}$ and $W_\gamma^{(1)}\subset$ some U_α of $\mathfrak{U}$ and, moreover, $\underset{\gamma\in\Gamma}{\smile} W_\gamma^{(1)}\supset G_1-A$.

By the same way we can have a locally finite collection $\{W_\delta^{(2)} \mid \delta\in\Delta\}$ of open sets of X such that for each $\delta\in\Delta$, $W_\delta^{(2)}\subset G_2-G_1\smile\bar{B}$ and $W_\delta^{(2)}\subset$ some U_α of $\mathfrak{U}$ and, moreover, $G_2-G_1\smile A\subset\underset{\delta\in\Delta}{\smile} W_\delta^{(2)}$.

Now, if we let $\mathfrak{V}=\{V_\beta, W_\gamma^{(1)}, W_\delta^{(2)} \mid \beta\in\Lambda, \gamma\in\Gamma, \delta\in\Delta\}$, then $\mathfrak{V}$ is a locally finite open covering to X which refines $\mathfrak{U}$. This shows that X is paracompact.

Lemma 2. *Let $\{G_1, G_2\}$ be an open covering of a regular T_1 space X. If each $\mathrm{Fr}(G_i)$ $(i=1,2)$ has the Lindelöf property, then there exist open sets $H_i(i=1,2)$ such that $H_i\supset\mathrm{Fr}(G_i)$ $(i=1,2)$, $H_1\frown H_2=\phi$ and $H_i\subset G_j$ $(i\neq j;\ i,j=1,2)$.*

Proof. Let $F_i=\mathrm{Fr}(G_i)$ $(i=1,2)$. Then F_i are mutually disjoint closed sets in X. Since X is regular and $F_i(i=1,2)$ have the Lindelöf property, there exist open sets $\{U_k^{(i)} \mid k=1,2,\cdots;\ i=1,2\}$ of X such that $G_j\supset\overset{\infty}{\underset{k=1}{\smile}} U_k^{(i)}\supset F_i$ and $F_i\frown\bar{U}_k^{(j)}=\phi$ $(k=1,2,\cdots;\ i\neq j$ and $i,j=1,2)$. If we let $V_n^{(i)}=U_n^{(i)}-\underset{k\leq n}{\smile}\bar{U}_k^{(j)}$ $(i\neq j;\ i,j=1,2$ and $n=1,2,\cdots)$, then, by induction, for each positive integer n, m we get $V_n^{(i)}\frown V_m^{(j)}=\phi$ $(i\neq j;\ i,j=1,2)$. Now, if we put $H_i=\overset{\infty}{\underset{k=1}{\smile}} V_k^{(i)}$ $(i=1,2)$, then from $F_i\frown\bar{U}_k^{(j)}=\phi$ $(i\neq j;\ i,j=1,2)$ we get $H_i\supset F_i$ $(i=1,2)$ and $H_1\frown H_2=\phi$. This completes the proof.

Proof of Theorem 2. When we fix the one element G_α of $\mathfrak{U}$, then by the star-finiteness of $\mathfrak{U}$ only G_{β_i} $(\alpha\neq\beta_i;\ i=1,\cdots,n_\alpha)$ intersect $\bar{G}_\alpha$.

If we put $H_\alpha = \bigcup_{i=1}^{n_\alpha} G_{\beta_i} \cup G_\alpha$, then H_α contains $\overline{G}_\alpha$ and is paracompact as a subspace of X by Lemmas 1 and 2. Hence $\overline{G}_\alpha$ is paracompact. On the other hand, $\{\overline{G}_\alpha \mid \alpha \in \Omega\}$ is a locally finite closed covering of X; that is, if x is an arbitrary point of X, then, since $\mathfrak{U}$ is star-finite, there exists a neighbourhood $U(x)$ of x which intersects only a finite number of the elements of $\mathfrak{U}$. So $U(x)$ intersects only a finite number of the closures of the elements of $\mathfrak{U}$ and this shows that $\{\overline{G}_\alpha \mid \alpha \in \Omega\}$ is locally finite. This proves that X is paracompact by Theorem 1 (or [5]).

Proof of Theorem 3. Since $\mathrm{Fr}(G_\alpha)$ is compact, there is an open set H_α which is a union of a finite number of elements of $\mathfrak{U}$ and contains $\overline{G}_\alpha$. By Lemmas 1 and 2 H_α is paracompact as a subspace of X, so $\overline{G}_\alpha$ is paracompact. On the other hand, $\{\overline{G}_\alpha\}$ is locally finite. By Theorem 1 (or [5]) we get the paracompactness of X.

We have the following corollary of Theorems 2 and 3:

Corollary. *Let* $\{G_\alpha \mid \alpha \in \Omega\}$ *be a star-finite (locally finite) open covering of a regular space* X. *If each* G_α *is metrizable and each* $\mathrm{Fr}(G_\alpha)$ *has the Lindelöf property (compact), then* X *is metrizable.*

Proof. Since metrizable space is paracompact (see [7]), each G_α is paracompact. By Theorem 2 (or 3) we have that X is paracompact. Then there is a locally finite closed covering which refines $\{G_\alpha \mid \alpha \in \Omega\}$ and, so, by [6] X is metrizable.

Theorem 4. *Let* $\{G_n \mid n=1, 2, \cdots\}$ *is a countable open covering of a regular* T_1 *space* X. *If each* G_n *is paracompact as a subspace of* X *and each* $\mathrm{Fr}(G_n)$ *is compact, then* X *is paracompact.*

Proof. Let $\mathfrak{U} = \{U_\alpha \mid \alpha \in \Omega\}$ be an arbitrary open covering of X. Since $\mathrm{Fr}(G_n)$ is compact, there is a finite subcollection $\mathfrak{U}^{(n)} = \{U_{\alpha_{nj}} \mid j=1, \cdots, i_n\}$ of $\mathfrak{U}$ whose sum covers $\mathrm{Fr}(G_n)$. As X is regular and $\mathrm{Fr}(G_n)$ is compact, there are open sets H_n, H_n' of X such that $\mathrm{Fr}(G_n) \subset H_n \subset \overline{H}_n \subset H_n' \subset \overline{H}_n' \subset \bigcup_{j=1}^{i_n} U_{\alpha_{nj}}$. Since $G_n - H_n'$ is closed in G_n and G_n is collectionwise normal, there exists a locally finite collection $\mathfrak{B}^{(n)} = \{V_\beta^{(n)} \mid \beta \in \Omega_n\}$ of open sets of G_n such that $\mathfrak{B}^{(n)}$ refines $\{U_\alpha \cap G_n \mid \alpha \in \Omega\}$ and $G_n - H_n' \subset \bigcup_{\beta \in \Omega_n} V_\beta^{(n)} \subset G_n - \overline{H}_n$. For each n, by $\bigcup_{\beta \in \Omega_n} V_\beta^{(n)} \cap \mathrm{Fr}(G_n) = \phi$, we see that $\mathfrak{B}^{(n)}$ is locally finite in X. Then it is easily seen that $\bigcup_{n=1}^{\infty} \mathfrak{B}^{(n)} \cup \bigcup_{n=1}^{\infty} \mathfrak{U}^{(n)}$ is a σ-locally finite open covering of X which refines $\mathfrak{U}$. This proves that X is paracompact (see [4]).

Remark. A. H. Stone proved in [8] that if a regular space S is the union of a sequence of open metrizable sets S_n $(n=1, 2, \cdots)$, each of which has a compact frontier, then S is metrizable. The above Stone's theorem is deduced as an immediate consequence of Theorem 4.

References

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