110. On the Boundedness of Solutions of Difference-Differential Equations

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Introduction. In their paper [1], R. Bellman and K. L. Cooke have defined a kernel function $K(t, s)$ which has been used to obtain several theorems concerning the stability and boundedness of solutions of difference-differential equations with perturbed terms.

In the present paper, we shall establish some theorems on the boundedness of solutions of difference-differential equations which are, in general, not linear.

1. For the sake of simplicity, we consider an equation

(1.1) $$x'(t)=A(t)x(t)+B(t)x(t-1)+w(t) \quad (0\leqq t<\infty)$$

under the conditions

(1.2) $$x(t-1)=\varphi(t) \quad (0\leqq t<1) \quad \text{and} \quad x(0)=x_0.$$

It is supposed that $A(t)$, $B(t)$, and $w(t)$ are continuous for $0\leqq t<\infty$, $\varphi(t)$ is continuous for $0\leqq t<1$, and $\lim_{t\to 1-0}\varphi(t)=\varphi(1-0)$ exists. Then, it is well known that there exists a unique solution of (1.1) under the initial conditions (1.2) for $0\leqq t<\infty$.

Now, we define a transformation

(1.3) $$y(t)=\begin{cases} x(t)-\varphi(t+1) & (-1\leqq t<0), \\ x(t)-x_0 & (0\leqq t<\infty). \end{cases}$$

Then, by (1.3), (1.1) is reduced to the equation with respect to y, that is,

(1.4) $$y'(t)=A(t)y(t)+B(t)y(t-1)+w_1(t)$$

under the condition $y(t-1)\equiv 0$ $(0\leqq t\leqq 1)$, where $w_1(t)$ is as follows:

$$w_1(t)=\begin{cases} x_0A(t)+B(t)\varphi(t)+w(t) & (0\leqq t<1), \\ x_0A(t)+x_0B(t)+w(t) & (1\leqq t<\infty). \end{cases}$$

By using the same kernel function $K(t, s)$ as defined in [1], the unique solution $y=y(t)$ of (1.4) under the condition $y(t-1)\equiv 0$ on $0\leqq t\leqq 1$ is represented by the integral

(1.5) $$y(t)=\int_0^t K(t, s)w_1(s)ds \quad (0\leqq t<\infty).^{1)}$$

Thus, it follows from (1.3) that

(1.6) $$x(t)=x_0+\int_0^t K(t, s)w_1(s)ds \quad (0\leqq t<\infty).$$

1) The method to obtain (1.5) is just the same as in [1].

Especially, if $w(t)\equiv 0$ on $0\leqq t<\infty$ and $\varphi(t)\equiv 0$ on $0\leqq t<1$, the equation (1.6) leads us to

$$(1.7)\quad x(t)=\begin{cases} x_0\left(1+\int_0^t K(t,s)A(s)ds\right) \\ x_0\left(1+\int_0^1 K(t,s)A(s)ds+\int_0^t K(t,s)(A(s)+B(s)\right)ds & (1\leqq t<\infty). \end{cases}$$

2. Now, we consider a perturbed equation

$$(2.1)\qquad x'(t)=A(t)x(t)+B(t)x(t-1)+f(t, x(t), x(t-1))$$

for $0\leqq t<\infty$ under the conditions

$$(2.2)\qquad x(t-1)=\varphi(t)\quad (0\leqq t<1)\quad \text{and}\quad x(0)=x_0.$$

The kernel function for the equation

$$(2.3)\qquad x'(t)=A(t)x(t)+B(t)x(t-1)$$

will be denoted by $K(t,s)$. It is supposed that the existence and uniqueness of the solution of (2.1) with (2.2) are guaranteed for $0\leqq t<\infty$. Then the following theorem will be established.

THEOREM 1. *In the equation* (2.1) *we suppose that the following conditions are satisfied*:

(i) *the unique solution* $x_0(t)$ *of* (2.3) *with* (2.2) *is bounded*;[2)]

(ii) $f(t,x,y)$ *is continuous and*

$$(2.4)\qquad |f(t,x,y)|\leqq h(t)(|x|+|y|)$$

for $0\leqq t<\infty$, $|x|<\infty$, $|y|<\infty$, *where* $h(t)$ *is continuous for* $0\leqq t<\infty$ *and*

$$(2.5)\qquad \int_0^\infty h(t)dt<\infty;$$

(iii) *the kernel function* $K(t,s)$ *is bounded, that is,*

$$(2.6)\qquad |K(t,s)|\leqq c\quad (0\leqq s\leqq t<\infty);$$

(iv) $\varphi(t)$ *is continuous for* $0\leqq t<1$, *and* $\lim_{t\to 1-0}\varphi(t)$ *exists.*

Then, the solution of (2.1) *with* (2.2) *is bounded for* $0\leqq t<\infty$.[3)]

PROOF. By means of the kernel function $K(t,s)$, it follows from (1.5) that the solution of (2.1) with (2.2) is represented by

$$x(t)=x_0(t)+\int_0^t K(t,s)f(s,x(s),\ x(s-1))ds.$$

Now we have to consider two cases:

I. The case $0\leqq t\leqq 1$. It follows from (2.2), (2.4), and (2.6) that

$$|x(t)|\leqq|x_0(t)|+\int_0^t|K(t,s)|\,|f(s,x(s),\ \varphi(s))|\,ds$$
$$\leqq c_1+c\int_0^t h(s)(|x(s)|+|\varphi(s)|)\,ds$$

2) A sufficient condition that the hypothesis (i) is satisfied is that $A(t)$ and $B(t)$ are absolutely integrable for $0\leqq t<\infty$, which will be established in Theorem 3.

3) Here, the upper bound of $|x(t)|$ may depend on x_0 and $\varphi(t)$.

$$\leqq c_2+c\int_0^t h(s)\,|x(s)|\,ds,$$

where c_1 is the upper bound for $|x_0(t)|$ and

$$c_2=c_1+c\int_0^1 h(s)\,|\varphi(s)|\,ds.$$

This inequality leads us to

$$|x(t)|\leqq c_2 \exp\left(c\int_0^t h(s)ds\right)\leqq c_2 \exp\left(c\int_0^\infty h(s)ds\right),$$

which implies that $|x(t)|$ is bounded.

II. The case $1\leqq t<\infty$. It follows by (2.2), (2.4), and (2.6) that

$$|x(t)|\leqq|x_0(t)|+\int_0^1 |K(t,s)|\,|f(s,x(s),\ \varphi(s))|\,ds$$

$$+\int_1^t |K(t,s)|\,|f(s,x(s),\ x(s-1))|\,ds$$

$$\leqq c_2+c\int_0^t (h(s)+h(s+1))\,|x(s)|\,ds.$$

This inequality leads us to

$$|x(t)|\leqq c_2 \exp\left(c\int_0^t (h(s)+h(s+1))\,ds\right)\leqq c_2 \exp\left(2c\int_0^\infty h(s)ds\right),$$

which implies the boundedness of $|x(t)|$.

3. We shall now establish another boundedness theorem without using any kernel functions. The equation to be discussed here is as follows:

(3.1) $$x'(t)=f(t,x(t),\ x(t-1)) \quad (0\leqq t<\infty)$$

under the initial conditions

(3.2) $$x(t-1)=\varphi(t) \quad (0\leqq t<1) \quad \text{and} \quad x(0)=x_0,$$

where $\varphi(t)$ is a function the same as before. It is supposed that the existence of solutions for $0\leqq t<\infty$ is guaranteed.

THEOREM 2. *We suppose that in the equation* (3.1) *with* (3.2), $f(t,x,y)$ *satisfies the following conditions*:

(i) $f(t,x,y)$ *is continuous for* $0\leqq t<\infty$, $|x|<\infty$, $|y|<\infty$;

(ii)

(3.3) $$|f(t,x,y)|\leqq h(t)(|x|+|y|)$$

for $0\leqq t<\infty$, $|x|<\infty$, $|y|<\infty$;

(iii) $h(t)$ *is continuous for* $0\leqq t<\infty$ *and*

(3.4) $$\int_0^\infty h(t)dt<\infty.$$

Then, any solution of (3.1) *with* (3.2) *is bounded for* $0\leqq t<\infty$.

PROOF. Let $x=x(t)$ be a solution of (3.1) with (3.2). Then, by means of the initial condition $x(0)=x_0$, it follows from (3.1) that

$$x(t)=x_0+\int_0^t f(s, x(s),\ x(s-1))ds \quad (0\leqq t<\infty). \tag{3.5}$$

I. The case $0\leqq t\leqq 1$. It follows from (3.2), (3.3), (3.5) that

$$\begin{aligned}|x(t)|&\leqq|x_0|+\int_0^t|f(s, x(s),\ \varphi(s))|ds\\&\leqq|x_0|+\int_0^t h(s)(|x(s)|+|\varphi(s)|)ds\\&\leqq c_3+\int_0^t h(s)|x(s)|ds,\end{aligned}$$

where

$$c_3=|x_0|+\int_0^1 h(s)|\varphi(s)|ds,$$

which leads us to the inequality

$$|x(t)|\leqq c_3\exp\left(\int_0^t h(s)ds\right)\leqq c_3\exp\left(\int_0^\infty h(s)ds\right). \tag{3.6}$$

II. The case $1\leqq t<\infty$. It follows from (3.2), (3.3), (3.5) that

$$\begin{aligned}|x(t)|&\leqq|x_0|+\int_0^1|f(s, x(s),\ \varphi(s))|ds+\int_1^t|f(s, x(s),\ x(s-1))|ds\\&\leqq|x_0|+\int_0^1 h(s)(|x(s)|+|\varphi(s)|)ds+\int_1^t h(s)(|x(s)|+|x(s-1)|)ds\\&\leqq c_3+\int_0^t (h(s)+h(s+1))|x(s)|ds,\end{aligned}$$

which leads us to the inequality

$$|x(t)|\leqq c_3\exp\left(\int_0^t(h(s)+h(s+1))ds\right)\leqq c_3\exp\left(2\int_0^\infty h(s)ds\right), \tag{3.7}$$

which implies together with (3.6) the boundedness of $|x(t)|$.

It is to be noted that the inequalities (3.6) and (3.7) show us not only the boundedness but also the stability of solutions, provided that $|x_0|$ and $|\varphi(t)|$ are sufficiently small.

4. As for difference-differential equations of neutral type, we shall establish a boundedness theorem, for which the equation to be discussed here is

$$x'(t)=f(t,x(t),\ x(t-1),\ x'(t-1)) \tag{4.1}$$

under the initial conditions

$$x(t-1)=\varphi(t)(0\leqq t<1) \quad\text{and}\quad x(0)=x_0, \tag{4.2}$$

where $f(t,x,y,z)$ is continuous and bounded, $|f(t,x,y,z)|\leqq M$, for $0\leqq t<\infty$, $|x|<\infty$, $|y|<\infty$, $|z|<\infty$, and $\varphi(t)$ is a given function as before, continuously differentiable for $0<t<1$, $\lim_{t\to 1-0}\varphi'(t)$, $\lim_{t\to +0}\varphi'(t)$ exist.

It is supposed that the existence and uniqueness theorems are guaranteed for $0\leqq t<\infty$.

THEOREM 3. *In the equation* (4.1) *with* (4.2) *we suppose that the following conditions are satisfied:*

(i) $|f(t, x, y, z)| \leqq h(t)(|x|+|y|+|z|)$ *for* $0 \leqq t < \infty$, $|x| < \infty$, $|y| < \infty$, $|z| < \infty$;

(ii) $$\int_0^\infty h(t)dt < \infty.$$

Then, the unique solution of (4.1) *with* (4.2) *is bounded for* $0 \leqq t < \infty$.

PROOF. I. The case $0 \leqq t \leqq 1$. If follows from (4.2) and (i), (ii) that

$$\begin{aligned}|x(t)| &\leqq |x_0| + \int_0^t |f(s, x(s), \varphi(s), \varphi'(s))| ds \\ &\leqq |x_0| + \int_0^t h(s)(|x(s)| + |\varphi(s)| + |\varphi'(s)|) ds \\ &\leqq c_4 + \int_0^t h(s)|x(s)| ds,\end{aligned}$$

where

$$c_4 = |x_0| + \int_0^1 h(s)(|\varphi(s)| + |\varphi'(s)|) ds.$$

II. The case $1 \leqq t < \infty$. It follows from (4.2) and (i), (ii) that

$$\begin{aligned}|x(t)| &\leqq |x_0| + \int_0^1 |f(s, x(s), \varphi(s), \varphi'(s))| ds + \int_1^t |f(s, x(s), x(s-1), x'(s-1))| ds \\ &\leqq |x_0| + \int_0^1 h(s)(|x(s)| + |\varphi(s)| + |\varphi'(s)|) ds \\ &\quad + \int_1^t h(s)(|x(s)| + |x(s-1)| + |x'(s-1)|) ds.\end{aligned}$$

Since $|x'(t)| \leqq M$, we have

$$u(t) \leqq c_5 + \int_0^t (h(s) + h(s+1)) u(s) ds,$$

where $u(t) = |x(t)| + |x'(t)|$ and $c_5 = c_4 + M$. Then it follows that

$$|x(t)| \leqq c_6 \exp\left(2\int_0^\infty h(s)ds\right) \quad (0 \leqq t < \infty),$$

where $c_6 = \text{Max}(c_4, c_5)$, which implies the boundedness of $|x(t)|$.

Reference

[1] R. Bellman and K. L. Cooke: Stability theory and adjoint operators for linear differential-difference equations, Trans. Amer. Math. Soc., **92**, 470-500 (1959).