143. Note on H-spaces

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1. Let G be a topological group and also a CW-complex (where the connectedness is not assumed), and $p: E\to B$ be a universal bundle with group G, where "universal" means that all the homotopy groups of E vanish. (The existence of a universal bundle is proved by Milnor [1].) Let ΩB be a loop space in B with base point $*=p(G)$ and 0_* the constant loop.

The following theorem is due to Samelson [4], essentially.

Theorem 1. *There is an H-homomorphism* $f: G\to \Omega B$, $f(e)=0_*$ (e *the unit of* G), *which is also a weak homotopy equivalence.*

Here, "H-homomorphism" means that the two maps

$$(x, y)\to f(xy) \quad\text{and}\quad (x, y)\to f(x)\circ f(y),$$

of $(G\times G, (e, e))$ into $(\Omega B, 0_*)$, are homotopic; and "weak homotopy equivalence" means that f induces isomorphisms of all the homotopy groups of G and ΩB, i.e., more precisely speaking,

(a) $f_*: \pi_0(G)\to\pi_0(\Omega B)$ is $1-1$ and onto, and

(b) $(f\,|\,C(G, x))_*: \pi_i(C(G, x))\to\pi_i(C(\Omega B, f(x)))$ is an isomorphism for any $x\in G$ and positive integer i, where $C(X, x)$ is the arcwise-connected component of X containing $x\in X$.

Proof. Because G is a CW-complex and $\pi_i(E)=0$ for $i\geqq 0$, G is contractible in E to e, leaving e fixed. Denote such a contraction by

$$k_t: G\to E,\ 0\leqq t\leqq 1:\ k_0=\text{identity},\ k_1(G)=k_t(e)=e.$$

The map $f: G\to \Omega B$ is defined by

$$f(x)(t)=p\circ k_t(x), \quad \text{for } x\in G.$$

By the same proof of [4, Theorem I], it is proved that f is an H-homomorphism, noticing that $G\times G$ has the same homotopy type of a CW-complex [2, Proposition 3], and a map of a CW-complex into E is homotopic to the constant map.

Consider the diagram

$$\begin{array}{ccc} \pi_i(E, G, e) & \xrightarrow{\partial} & \pi_{i-1}(G, e) \\ \downarrow p_* & & \downarrow f_* \\ \pi_i(B, *) & \xrightarrow{T} & \pi_{i-1}(\Omega B, 0_*), \end{array}$$

where T is the natural isomorphism. p_*, T and ∂ are isomorphisms for $i\geqq 2$ and $1-1$, onto for $i=1$.

For a map $h: (I^{i-1}, \dot{I}^{i-1})\to(G, e)$, define $\bar{h}: (I^i, \dot{I}^i, J^{i-1})\to(E, G, e)$ by $\bar{h}(s, t)=k_t\circ h(s)$ for $(s, t)\in I^{i-1}\times I$. Then

$$(T(p\circ\bar{h})(s))(t)=p\circ\bar{h}(s,t)=p\circ k_t\circ h(s)=((f\circ h)(s))(t).$$

This shows the commutativity of the above diagram, and so we obtain (a) and a part of (b):

$$(f|C(G,e))_*\colon \pi_i(C(G,e))\approx\pi_i(C(\Omega B,0_*)).$$

For any $x\in G$, the diagram

$$\begin{array}{ccc} C(G,e) & \xrightarrow{g} & C(G,x) \\ \downarrow f & & \downarrow f \\ C(\Omega B,0_*) & \xrightarrow{g} & C(\Omega B,f(x)), \end{array} \qquad \begin{array}{l} .g(y)=yx, \\ g'(l)=l\circ f(x), \end{array}$$

is commutative, up to a homotopy, because f is an H-homomorphism; and g is a homeomorphism and g' a homotopy equivalence. Therefore, (b) follows from the special case $x=e$. q.e.d.

Theorem 2. *If B is an H-space, G is homotopy-commutative, i.e. the two maps:*

$$C_0, C_1\colon (G\times G,(e,e))\to(G,e),\ C_0(x,y)=xy,\ C_1(x,y)=yx,$$

are homotopic.

Proof. It is well known that ΩB is homotopy-commutative. Therefore, the two maps $f\circ C_0$, $f\circ C_1$: $(G\times G,(e,e))\to(\Omega B,0_*)$ are homotopic, since f is an H-homomorphism.

Also, f is a weak homotopy equivalence. Hence, it is easily proved that any two maps φ_0, φ_1 of a CW-complex into G are homotopic if $f\circ\varphi_0$, $f\circ\varphi_1$ are so, applying the usual technique on each connected component of a CW-complex. Therefore, C_0 and C_1 are homotopic, noticing that $G\times G$ is a CW-complex, up to a homotopy type. q.e.d.

By this theorem, it follows immediately

Theorem 3. *A classifying space $B_G=K(G,1)$ of a discrete group G is an H-space if and only if, G is abelian.*

2. Le $S(\infty)$ be the infinite symmetric group and μ: $S(\infty)\times S(\infty)\to S(\infty)$ be the homomorphism, defined by

$$\mu(\alpha,\beta)(2i-1)=2\alpha(i)-1,\ \mu(\alpha,\beta)(2i)=2\beta(i),$$

for positive integer i. Recently, Nakaoka [3] has proved that the homology $H_*(S(\infty);k)$, for a field k, is a Hopf algebra by the product induced by μ. The homomorphism μ induces a map $\bar{\mu}$: $B_{S(\infty)}\times B_{S(\infty)}\to B_{S(\infty)}$, where $B_{S(\infty)}$ is assumed to be a CW-complex, and $H_*(B_{S(\infty)};k)=H_*(S(\infty);k)$ is a Hopf algebra by the product induced by $\bar{\mu}$.

On the other hand, $B_{S(\infty)}$ is not an H-space, by Theorem 3. Therefore, $B_{S(\infty)}$ is an example of such an arcwise-connected space that its homology is a Hopf algebra, by the product induced by a map of spaces, but it is not an H-space.

Concerning to this situation, we have the following theorem, where the commutativity of the fundamental group is necessary by this example.

Theorem 4. *Let X be an arcwise-connected CW-complex such that $\pi_1(X)$ is abelian. Assume that there is a map μ: $X\times X\to X$ such* *that* $H_*(X; k)$ *is a Hopf algebra by the product induced by* μ. *Then,* X *is an H-space.*

Proof. Assume that $\mu(*, *)=*$ for a point $*\in X$. The maps $x\to\mu(x, *)$ and $x\to\mu(*, x)$, of X into itself, induce automorphisms of all the homology groups of X, and, hence, of all the homotopy groups, since $\pi_1(X)$ is abelian. Therefore, it is easy to prove that the map:

$$l_i: (X\times X, (*, *))\to(X\times X, (*, *)),\ l_i(x_1, x_2)=(\mu(x_1, x_2), x_i),$$

induces automorphisms of the homotopy groups of a CW-complex $X\times X$ up to a homotopy type, for $i=1, 2$.

Now, the conclusion of this theorem is a consequence of [5, Lemma 1.2]. q.e.d.

References

[1] John Milnor: Construction of universal bundles, II, Ann. of Math., **63**, 430-436 (1956).

[2] ——: On spaces having the homotopy type of a CW-complex, Trans. Amer. Math. Soc., **90**, 272-280 (1959).

[3] M. Nakaoka: Homology of infinite symmetric group (to appear).

[4] H. Samelson: Groups and spaces of loops, Comment. Math. Helv., **28**, 278-287 (1954).

[5] M. Sugawara; On the homotopy-commutativity of groups and loop spaces, Mem. Coll. Sci., Univ. Kyoto, ser. A, **33**, 257-269 (1960).