138. *Weakly Compact Operators on the Spaces of Continuous Functions*

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In this note we shall give a brief account of some properties of weakly compact operators on the spaces of continuous functions on general spaces. Our main purpose is to extend some results of Arens [1] and Grothendieck [5]. Full details will appear in Osaka Mathematical Journal.

1. Let E and F be locally convex linear topological spaces. Then a continuous linear operator T of E into F is said to be a *compact* (resp. *weakly compact*) *operator* if T maps a neighborhood of 0 in E into a compact (resp. weakly compact) subset in F. A completely regular Hausdorff space X is said to be a *k_0-space* if whenever $U\frown K$ is a neighborhood of x_0 in K for a subset U $(\ni x_0)$ and for any compact subset K $(\ni x_0)$, U is a neighborhood of x_0 in X. A neighborhood here need not be an open set. A k_0-space is a k-space,[1] and any completely regular space satisfying the 1st axiom of countability or any locally compact Hausdorff space is always ak_0-space. Let X be a topological space and $\mathfrak{S}$ be a set of compact subsets. Then we denote by $C_{\mathfrak{S}}(X)$ the space of all continuous functions on X with the topology of uniform convergence of sets in $\mathfrak{S}$. "$\bigcup\mathfrak{S}=X$" denotes that the sum of all subsets in $\mathfrak{S}$ is X.

We first extend a theorem of Bartle [2][2] to the case of locally convex topological linear spaces.

Theorem 1. (i) *Let E be a barrelled locally convex linear space. Let Y be a completely regular Hausdorff space and let $\mathfrak{S}$ be a set of compact sets in Y with $\bigcup\mathfrak{S}=Y$. Then a linear operator T of E into $C_{\mathfrak{S}}(Y)$ is continuous if and only if there is a continuous mapping τ of Y into E' with respect to the topology $\sigma(E',E)$ such that $(Te)y=\langle\tau y, e\rangle$ for any $e\in E$ and for any $y\in Y$.*

(ii) *Let E be a Banach space. Let Y be a completely regular Hausdorff space and $\mathfrak{S}$ be a set of compact subsets in Y with $\bigcup\mathfrak{S}=Y$. Then a continuous linear operator T of E into $C_{\mathfrak{S}}(Y)$ is weakly compact if and only if there is a continuous mapping τ of Y into E' with respect to the topology $\sigma(E',E'')$ such that $(Te)y=\langle\tau y, e\rangle$ for $e\in E$ and $y\in Y$.*

(iii) *Let E be a locally convex topological linear space. Let Y be*

1) Cf. for example, Kelly [6].
2) Cf. [2, p. 55, Theorem 10.2].

a k_0-space and let $\mathfrak{S}$ be the set of all compact subsets in Y. Then a continuous linear operator T of E into $C_{\mathfrak{S}}(Y)$ is compact if and only if there is a continuous mapping τ of Y into $E'_{\mathcal{E}}$[3] *for a balanced convex w^*-closed equicontinuous set $\mathcal{E}$ in E', and $(Te)y=\langle \tau y, e\rangle$ for $e\in E$ and $y\in Y$.*

For the proof of (iii) we need to establish the following generalized Ascoli's theorem.

Lemma (Ascoli). *Let X be a k_0-space and let $\mathfrak{S}$ be the set of all compact subsets in X. Then a set A in $C_{\mathfrak{S}}(X)$ is relative compact if and only if A is an equicontinuous set in $C_{\mathfrak{S}}(X)$ and $A(x)=\{f(x)\mid f\in A\}$ is bounded for any $x\in X$.*

Let X be a stonian space.[4] Then if a sequence $\{\mu_i\}\subset C(X)'$ converges to 0 with respect to the topology $\sigma(C(X)', C(X))$, then $\{\mu_i\}$ converges to 0 with respect to the topology $\sigma(C(X)', C(X)'')$.[5] Therefore, by Theorem 1, we have the following

Theorem 2. *Let X be a stonian space. Let Y be a completely regular Hausdorff space satisfying the 1st axiom of countability and let $\mathfrak{S}$ be a set of compact set in Y with $\bigcup\mathfrak{S}=Y$. Then any continuous linear operator of $C(X)$ into $C_{\mathfrak{S}}(Y)$ is weakly compact.*

Corollary 1 (*Grothendieck*). *Let X be a stonian space and let E be a separable*[6] *complete Hausdorff locally convex linear space. Then any continuous linear operator of $C(X)$ into E is weakly compact.*

Corollary 2. *Let X be an extremally disconnected space*[4] *and let $\mathfrak{S}$ be a non-empty set of compact sets in X. Let Y be a compact Hausdorff space satisfying the 1st axiom of countability. Then any continuous linear operator of $C_{\mathfrak{S}}(X)$ into $C(Y)$ is weakly compact.*

Remark. If a completely regular space satisfies the 1st axiom of countability, then $C_{\mathfrak{S}}(X)$ is, in general, not separable.

2. Let X be a metric space and let F be a closed subspace in X. Then there is for each f in $C_u(F)$[7] an element Tf in $C_u(X)$, with $(Tf)(x)=f(x)$ for all x in F, such that T is non-negative, linear isometry of $C_u(F)$ into $C_u(X)$ (the simultaneous extension theorem). But Day [4] gave an example of a compact Hausdorff space X and of a closed subspace F such that there is no linear mapping of $C_u(F)$ into $C_u(X)$ which is a simultaneous extension of all elements of $C_u(F)$. His

3) Let $\mathcal{E}$ be a w^*-closed balanced equicontinuous convex subset in E'. Then we denote by $E'_{\mathcal{E}}$ the normed space whose unit sphere is $\mathcal{E}$.

4) A completely regular Hausdorff space X is said to be extremally disconnected if for any open set U in X the closure $\bar{U}$ of U is also open. A compact Hausdorff space is stonian if it is extremally disconnected.

5) Cf. [5, p. 168, Theorem 9].

6) A topological space is separable if it contains a countable dense subset.

7) If X is a topological space, then we denote by $C_u(X)$ the Banach space of all bounded continuous functions on X with $\|f\|=\sup_{x\in X}|f(x)|$.

example is the following: Let X be the topological product space of the closed unit intervals I_λ $(\lambda \in \Lambda)$ and let the set Λ of indices be uncountable. Let S be the unit sphere of $l_q(\Lambda)$ with the topology $\sigma(l_q(\Lambda), l_p(\Lambda))$, where $p>1$, $q>1$ and $p^{-1}+q^{-1}=1$. Then we can regard S as a closed subset in X and $l_p(\Lambda)$ as a linear subspace of $C(S)$. Day showed that there is no continuous linear operator T from $L=l_p(\Lambda)$ into $C_u(X)$ such that, for each f in L, Tf is an extension of f. We see that the space L is generated by a weakly compact set in $C_u(S)$. By Theorem 1 and Theorem of Arens,[8] we have the following

Theorem 3. *Let X be a paracompact space and let F be a closed subset in X. Let L be a closed linear subspace in $C_u(F)$ which is generated by a compact set in $C_u(F)$. Then there is a simultaneous extension of L into $C_u(X)$.*

Corollary (Arens). *Let X be a paracompact space and let F be a closed subset in X. Let X be a separable closed linear subspace in $C_u(F)$. Then there is a simultaneous extension of L into $C_u(X)$.*

3. We next extend a theorem of Bartle, Dunford and J. Schwartz[9] to the case of locally convex topological linear spaces.

Theorem 4. (a) *Let X, Y be completely regular Hausdorff spaces and let Y be a hemi-compact*[10] *k-space. Let $\mathfrak{S}$ be the set of all compact subsets in X and let $\mathfrak{T}$ be the set of all compact subsets in Y. Then a continuous linear operator T of $C_{\mathfrak{S}}(X)$ into $C_{\mathfrak{T}}(Y)$ is weakly compact if and only if there are a kernel function $k(x, y)$ on $K\times Y$ (for some $K\in\mathfrak{S}$) and a non-negative Borel measure ν on K such that*

$$(*)\qquad (Tf)y=\int (f\,|\,K)(x)k(x, y)\nu(dx)$$

and k satisfies the conditions:

(i) *for any $y\in Y$, $k(x, y)\in L^1(K, \nu)$,*

(ii) *for any Borel set E in K, $\int_E k(x, y)\nu(dx)$ is a continuous function on Y,*

(iii) *for any $H\in\mathfrak{T}$, $\sup_{y\in H}\int |k(x, y)|\nu(dx)<+\infty$.*

(b) *Let X be a completely regular Hausdorff space and let Y be a hemi-compact k_0-space. Let $\mathfrak{S}$ be the set of all compact subsets of X and let $\mathfrak{T}$ be the set of all compact subsets in Y. Then a continuous linear operator T of $C_{\mathfrak{S}}(X)$ into $C_{\mathfrak{S}}(Y)$ is compact if and only if there is a kernel function $k(x, y)$ on $K\times Y$ (for some $K\in\mathfrak{S}$) and non-negative Borel measure ν on K such that the equation $(*)$ is satis-*

8) Cf. [1, p. 18, Theorem 4.1].

9) Cf. [3, Theorems 4.3 and 4.4].

10) A topological space X is said to be hemi-compact if there is a sequence $\{K_n\}$ of compact subsets in X such that $X=\bigcup_{n=1}^{\infty} K_n$ and $K\subset$ some K_n for any compact set K in X.

fied and k satisfies the condition (i) *and*

(iv) *if $y_\lambda \to y_0$ in Y, then*

$$\lim_{y_\lambda \to y_0} \int |k(x, y_\lambda) - k(x, y_0)| \nu(dx) = 0.$$

Let J be a set of indices. Then we denote by $m(J)$ the space of all bounded real functions on J with $\|x\| = \sup_{j \in J} |x(j)|$, and denote by $c_0(J)$ the subspace of those x in $m(J)$ for which for each $\varepsilon > 0$ the set of j with $|x(j)| > \varepsilon$ is finite; that is $c_0(J)$ is the set of functions vanishing at infinity on the discrete space J. Now, we consider a compact space such that all points are isolated except one point. Then the conditions (ii) and (iii) of Theorem 4 imply the condition (iv) if Y satisfies the 1st axiom of countability, and we have the following

Theorem 5. *Let E be a separable metrizable locally convex linear space. Then any weakly compact linear operator of $c_0(J)$ into E is compact.*

Corollary 1 (Grothendieck). *Any weakly compact linear operator of c_0 into a locally convex Hausdorff topological linear space is compact.*

Corollary 2. *Let E be a Banach space whose dual E' is separable. Then any continuous linear operator of the space m into E' is compact.*

Added in proof: Theorem 3 is not a proper extension of Corollary of Theorem 3. But this theorem can be extended under some conditions to the case of locally convex linear spaces.

References

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