153. *Necessary Results for Computation of Cyclic Parts in Steinhaus Problem*

By Kiyoshi ISÉKI

(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1960)

In my recent paper [1], we considered a problem of arithmetic. As the problem was stated in H. Steinhaus, Sto Zadan (in Polish), we call it *Steinhaus problem for power k*. We decided all cyclic parts of Steinhaus problem for power 3 (see may paper [1]).

Let α be a natural number, then α is expressed in the decimal system:

$$\alpha=10^{n-1}a_n+10^{n-2}a_{n-1}+\cdots+10^2a_3+10a_2+a_1.$$

For $k=3$, let α_1 be $a_n^3+a_{n-1}^3+\cdots+a_3^3+a_2^3+a_1^3$, then

$$\alpha-\alpha_1=(10^{n-1}-a_n^2)a_n+(10^{n-2}-a_{n-1}^2)a_{n-1}+\cdots$$
$$+(10^2-a_3^2)a_3+(10-a_2^2)a_2+(1-a_1^2)a_1.$$

If $a_1=a_2=9$, then we have

$$(1-a_1^2)a_1=-720,$$
$$(10-a_2^2)a_2=-639,$$

therefore, for any a_1, a_2, we have

$$(1-a_1^2)a_1+(10-a_2^2)a_2\geq -1359.$$

Let $n\geq 4$, the function $(10^{n-1}-x^2)x$ of x is an increasing function on $0\leq x\leq 9$. Hence, $(10^{n-1}-x^2)x=1992$ for $n=4$, $x=2$, and we obtain

$$(10^{n-1}-a_n^2)a_n\geq 1992,$$
$$(10^{i-1}-a_i^2)a_i\geq 0 \quad (i=3, 4,\cdots, n-1).$$

This implies

$$\alpha-\alpha_1\geq 1992-1359=633$$

and we have

$$\alpha>\alpha-633\geq\alpha_1.$$

This means $\alpha>\alpha_1$ and the sequence $\alpha, \alpha_1, \alpha_2,\cdots$ for numbers greater than 2000 is strictly decreasing. Therefore we have a number less than 1999 for some term α_n. Hence it is sufficient to find *the cyclic parts of all numbers less than* 1999.

Such an estimate is possible for every k.

For Steinhaus problem for $k=4$, from

$$\alpha-\alpha_1=(10^{n-1}-a_n^3)a_n+\cdots+(10^2-a_3^3)a_3+(10-a_2^3)a_2+(1-a_1^3)a_1,$$

we have

$$(10^2-a_3^3)a_3+(10-a_2^3)a_2+(1-a_1^3)a_1$$
$$\geq -(629\times 9+719\times 9+728\times 9)=-18684,$$
$$(10^{i-1}-a_i^3)a_i\geq 0 \quad (i=4, 5,\cdots, n-1)$$

and, if $n=5$, $a_n\geq 2$, then

$$(10^4-a_5^3)a_5\geq 19984.$$

This implies

$$\alpha-\alpha_1 \geq 1300,$$

and $\alpha-1300 \geq \alpha_1$.

Then, for $k=4$ we must find *cyclic parts of numbers less than* 20000.

For $k=5$, the negative parts are greater than -226197. If $n=6$, $a_6 \geq 3$, then

$$(10^5-a_6^4)a_6 \geq 299757.$$

Therefore we have

$$\alpha-73560 > \alpha_1.$$

From the argument above, for $k=5$ we must calculate *the cyclic parts of numbers less than* 3×10^5.

As we pointed out in my paper [1], for $k=3$, the cyclic parts we can decide by the table of cyclic parts from 1 to 999.

Let us consider the numbers of the second column in the table. If the number α_1 for $\alpha=a_3a_2a_1$ is larger than 1000 (for example, $\alpha_1=1073$ for $\alpha=179$), for $1a_3a_2a_1$ $(=1000+a_3a_2a_1)$, we have $\alpha_1=1+a_3^3+a_2^3+a_1^3$ (for 1179, we have 1074). If α_1 contain the digit 0, we need not consider the number $1a_3a_2a_1$. Therefore, by the table in my paper [1], we need only consider: 1243 for 1189, 1467 for 1299, 1269 for 1389, 1486 for 1399, 1523 for 1499, 1198 for 1579, 1367 for 1589, 1584 for 1599, 1162 for 1669, 1289 for 1679, 1241 for 1688, 1458 for 1689, 1675 for 1699, 1199 for 1778, 1416 for 1779, 1368 for 1788, 1585 for 1789, 1537 for 1888, 1754 for 1889, 1971 for 1899, and 2188 for 1999.

We can omit the calculation of α_2, when it is less than 1000. Then we have the following list:

α	α_1	α_2	α_3			
1579	1198	1243	100	**1**		
1679	1289	1250	(125)	**371**		
1778	1199	1460	(146)	**371**		
1899	1971	1074	(147)	**153**		
1999	2188	1033	(133)	**250**	**217**	**352**

Therefore, we have the complete solution of Steinhaus problem for $k=3$.

In our later paper, we shall use the results mentioned above to decide the cyclic parts for power 4, 5.

Reference

[1] K. Iséki: A problem of number theory, Proc. Japan Acad., **36**, 578–583 (1960).