149. *On Relative Derivation of Additive Set-Functions*

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(Comm. by Z. SUETUNA, M.J.A., Dec. 12, 1960)

1. **Introduction.** This is a continuation of the paper [1] by the same author. Utilizing the extension of the Vitali covering theorem established there, we shall obtain in what follows a series of consequent theorems concerning relative derivation of additive set-functions. The greater part of our results will, however, be analogues of the corresponding results contained in the fourth chapter of Saks [2] (we shall henceforth quote this book simply as Saks for short). We shall omit the proofs whenever they are mere repetitions, at most with slight modifications, of those given in Saks. We remark explicitly that, as in [1], the space that underlies all our considerations is the real line $\boldsymbol{R}$.

2. **Preliminary remarks.** We shall be interested in finite real set-functions which are defined and completely additive on the class of the bounded Borel subsets of the real line $\boldsymbol{R}$. For brevity they will simply be called *additive set-functions*.

Suppose we are given a nonnegative additive set-function μ. We shall keep this notation fixed throughout the rest of the paper. As is well known (*vide* p. 71 of Saks), we can attach to μ a finite nonnegative additive interval-function F, defined for linear closed intervals and such that $F^*(X)=\mu(X)$ for all bounded Borel sets X in $\boldsymbol{R}$, where F^* means the outer measure of Carathéodory induced by F.

Now simple examples show that in general the interval-function F is not uniquely determined by μ. However, the outer measure F^* itself is uniquely determined by μ. To see this we observe firstly that for any set E (bounded or not) the value of $F^*(E)$ is equal to the infimum of $F^*(D)$ for open sets $D\supset E$, as stated on p. 68 of Saks. But here $F^*(D)$ must be independent of the choice of F, since F^* and μ coincide for bounded Borel sets and since D can be expressed as the limit of an ascending sequence of bounded open sets.

We are thus entitled to write henceforth $\tilde{\mu}$ for the function F^*, and all the results given in Saks on outer measures of Carathéodory and on measures induced by nonnegative additive interval-functions, as well as our extension of the Vitali theorem, will hold good for $\tilde{\mu}$. Moreover, it follows from what has been said above that we may also construct the function $\tilde{\mu}$ as follows: the value $\tilde{\mu}(D)$ for an open set D is by definition the supremum of the values of μ for bounded open sets in D, and we define $\tilde{\mu}(E)$ for an arbitrary set E as the infimum of $\tilde{\mu}(D)$ for open sets D containing E. Thus $\tilde{\mu}$ is completely determined when the values of μ for bounded open sets are known.

3. Relative derivates of additive set-functions. Given an additive set-function Φ we call *upper μ-derivate* of Φ at a point $c \in \boldsymbol{R}$ the upper limit of the ratio $\Phi(I)/\mu(I)$, where I is any closed interval containing c, whose length tends to zero. This μ-derivate will be denoted by $(\mu)\overline{\Phi}(c)$. To avoid any misunderstanding, it may be observed that we mean by the quotient $a/0$, where a is a finite real number, the values $+\infty$, 0, $-\infty$ according as $a>0$, $a=0$, $a<0$.

The *lower μ-derivate* $(\mu)\underline{\Phi}(c)$ has corresponding definition, and if the numbers $(\mu)\overline{\Phi}(c)$ and $(\mu)\underline{\Phi}(c)$ are equal, their common value is termed *μ-derivative* of Φ at the point c and will be denoted by $(\mu)\Phi'(c)$. If further $(\mu)\Phi'(c)$ is finite, the function Φ is said to be *μ-derivable* at c. The μ-derivates and the μ-derivative thus defined will be called to be *relative* to the function μ, in contradistinction to the usual derivates and derivative. Sometimes $(\mu)\Phi'(c)$ is termed *unique μ-derivate*, while $(\mu)\overline{\Phi}(c)$ and $(\mu)\underline{\Phi}(c)$ are termed *extreme μ-derivates.*

Needless to say, all the above definitions will remain quite meaningful when we replace the set-function Φ by any interval-function F defined for the linear closed intervals and assuming finite real values, and it is easy to see that the two extreme μ-derivates $(\mu)\overline{F}(x)$ and $(\mu)\underline{F}(x)$ are B-measurable on $\boldsymbol{R}$. In particular, *the upper and lower μ-derivates of an additive set-function are always B-measurable.*

4. Lebesgue's theorem on relative derivation. LEMMA. *If for a nonnegative additive set-function Φ and a finite real number a the inequality $(\mu)\overline{\Phi}(x)\geqq a$ holds at every point x of a bounded set A, then $\Phi(X)\geqq a\tilde{\mu}(A)$ holds for every bounded Borel set X containing A.*

LEBESGUE'S THEOREM. *Any additive set-function is μ-derivable almost everywhere $(\tilde{\mu})$.*

THEOREM. *If an additive set-function Φ is the limit of a monotone sequence $\Phi_1, \Phi_2, \cdots$ of additive set-functions, then we have $(\mu)\Phi'(x) = \lim_n (\mu)\Phi'_n(x)$ almost everywhere $(\tilde{\mu})$.*

5. Relative derivation of the indefinite integral. It is convenient to begin with a few definitions. By a *μ-lacunar point* of the real line we shall understand any point which can be enclosed in a closed interval I such that $\mu(I)=0$. Of course we may simply say "lacunar point" when there is no mistaking the meaning. As we shall see below, $\tilde{\mu}$-almost every point of $\boldsymbol{R}$ is nonlacunar.

Given a set A let us write $\mu_A(X)=\tilde{\mu}(AX)$ for bounded Borel sets X. Considered *qua* function of X, μ_A is clearly nonnegative and additive. The upper and lower μ-derivates of μ_A at a point x will be called respectively the *outer upper* and *outer lower μ-density* of A at x. The points at which these two densities are equal to 1 are termed *points of outer μ-density*, and the points at which they are equal to zero, *points of μ-dispersion*, for the set A. If the set A is measurable ($\tilde{\mu}$), we suppress the word "outer" in these expressions. In this latter case any point of μ-density for A is evidently a point of μ-dispersion for $\boldsymbol{R}-A$, and the converse is also true provided, however, that the point under consideration is nonlacunar (μ).

LEMMA. *The μ-lacunar points of the real line form together a Borel set of measure zero ($\tilde{\mu}$).*

PROOF. Let us write E for the set of the lacunar points. Supposing E nonvoid, we find at once that the family $\mathfrak{M}$ of all closed intervals for which μ vanishes is nonvoid and covers E in the Vitali sense. Therefore, by our extension of Vitali's theorem, we can extract from $\mathfrak{M}$ a disjoint (finite or infinite) sequence $I_1, I_2, \cdots$ of intervals so as to cover the set E almost entirely ($\tilde{\mu}$). We thus get

$$\tilde{\mu}(E) \leqq \tilde{\mu}(\bigcup_n I_n) = \sum_n \mu(I_n) = 0,$$

and this implies that $\tilde{\mu}(E)=0$. Moreover E must be a Borel set since it plainly coincides with the join of the family $\mathfrak{M}$.

THEOREM OF μ-DENSITY. *Given an arbitrary set A of real numbers, $\tilde{\mu}$-almost all the points of A are points of outer μ-density for A; and if further the set A is measurable ($\tilde{\mu}$), then $\tilde{\mu}$-almost all the points of $\boldsymbol{R}-A$ are points of μ-dispersion for A.*

The proof for this proceeds in essentially the same way as for the theorem on p. 117 of Saks, only that we make use of the lemma established just now.

THEOREM. *If an additive set-function Φ is the indefinite integral ($\tilde{\mu}$) of an extended-real function f which is defined on $\boldsymbol{R}$ and integrable ($\tilde{\mu}$) over bounded Borel sets, then $(\mu)\Phi'(x)=f(x)$ at $\tilde{\mu}$-almost all points x of $\boldsymbol{R}$.*

REMARK. Here and subsequently, integrals will only be taken over bounded Borel sets, and the additive class of sets which underlies the integration will be that of the Borel sets or of the sets measurable (Γ) according as the integration is performed, respectively, with reference to a nonnegative additive set-function or to an outer Carathéodory measure Γ. In relation to this it is easy to see that, when the function f of the theorem is B-measurable, the integrability ($\tilde{\mu}$) of f is equivalent on any bounded Borel set X to its integrability (μ) and that, further, the integral of f over X is the same with reference to $\tilde{\mu}$ as to μ.

6. Relative approximate continuity. Let $f(x)$ denote in this section an extended-real function defined on $\boldsymbol{R}$. We call $f(x)$ *approximately continuous* (μ) at a point c, if (i) there exists a set E measurable $(\tilde{\mu})$ which contains c and for which c is a point of μ-density, and (ii) the restriction of f to the set E is continuous at c.

THEOREM. *If f is measurable $(\tilde{\mu})$ on $\boldsymbol{R}$ and finite almost everywhere $(\tilde{\mu})$ on a set M measurable $(\tilde{\mu})$, then the function f is approximately continuous (μ) at $\tilde{\mu}$-almost all points of M.*

7. The Lebesgue decomposition. For the meanings of the expressions "absolutely continuous (μ) on E" and "singular (μ) on E", where E is a bounded Borel set, we refer the reader to Saks (p. 30, below). In each of these expressions, the reference to the set E will be omitted when the relevant property holds for all E. Thus, for example, an additive set-function Φ is absolutely continuous (μ) if, and only if, $\Phi(X)=0$ whenever X is a bounded Borel set such that $\mu(X)=0$.

LEMMA. *If an additive set-function Φ is singular (μ), then $(\mu)\Phi'(x)=0$ almost everywhere $(\tilde{\mu})$.*

LEBESGUE'S DECOMPOSITION THEOREM. *If Φ is an additive set-function, the μ-derivative $(\mu)\Phi'(x)$ is a B-measurable function which is integrable (μ) over bounded Borel sets, and the function Φ is expressible as the sum of its function of singularities (μ) and of the indefinite integral (μ) of its μ-derivative.*

Moreover, if in particular the function Φ is nonnegative, we have

$$\Phi(E)\geqq\int_E(\mu)\Phi'(x)d\mu(x)$$

for each bounded Borel set E, the equality sign holding when, and only when, Φ is absolutely continuous (μ) on E.

COROLLARIES. 1^0 *An additive set-function is absolutely continuous (μ) if, and only if, it is the indefinite integral (μ) of its μ-derivative.*

2^0 *An additive set-function is singular (μ) if, and only if, its μ-derivative vanishes almost everywhere $(\tilde{\mu})$.*

3^0 *If W denotes the absolute variation of an additive set-function Φ, then $(\mu)W'(x)=|(\mu)\Phi'(x)|$ almost everywhere $(\tilde{\mu})$.*

8. De la Vallée Poussin's decomposition. We shall now prove a theorem which corresponds to the theorem of de la Vallée Poussin stated on p. 127 of Saks and constitutes the principal result of this paper. The proof given below will be quite different from that adopted in Saks; above all we shall not utilize anything like Tonelli's theorem on the length of parametric curves, so that our proof will be somewhat simpler in idea.

DE LA VALLÉE POUSSIN'S DECOMPOSITION THEOREM. *Let Φ be an additive set-function, with W for its absolute variation. Then*

(i) *the points at which Φ has no μ-derivative finite or infinite* *form a Borel set for which both $\tilde{\mu}$ and $\tilde{W}$ vanish.*

(ii) *If $E_{+\infty}$ and $E_{-\infty}$ denote the Borel sets of the points x at which the μ-derivative $(\mu)\Phi'(x)$ is equal to $+\infty$ and to $-\infty$ respectively, then for any bounded Borel set X we have the formulas:*

$$\Phi(X)=\Phi(XE_{+\infty})+\Phi(XE_{-\infty})+\int_X (\mu)\Phi'(x)d\mu(x), \tag{1}$$

$$W(X)=\Phi(XE_{+\infty})-\Phi(XE_{-\infty})+\int_X |(\mu)\Phi'(x)|d\mu(x). \tag{2}$$

(iii) *The points x at which the two derivatives $(\mu)\Phi'(x)$ and $(\mu)W'(x)$ exist and fulfil the relation $(\mu)W'(x)=|(\mu)\Phi'(x)|$ constitute together a Borel set M such that $\tilde{\mu}(\boldsymbol{R}-M)=\tilde{W}(\boldsymbol{R}-M)=0$.*

PROOF. *re* (i): Let us write $\nu=\mu+W$ and let A denote the Borel set of the points nonlacunar (ν) at which the three functions μ, Φ, and W are all derivable (ν). Then $\tilde{\nu}(\boldsymbol{R}-A)=0$ on account of Lebesgue's theorem (§4) and the lemma of §5. Furthermore μ, Φ, and W are evidently all absolutely continuous (ν) and at every point x of A their ν-derivatives fulfil the relations

$$0\leqq(\nu)\mu'(x)\leqq 1 \quad \text{and} \quad |(\nu)\Phi'(x)|\leqq(\nu)W'(x)\leqq 1.$$

Now write B for the Borel set of the points x of A at each of which the ν-derivatives $(\nu)\mu'(x)$ and $(\nu)\Phi'(x)$ do not both vanish and so the μ-derivative $(\mu)\Phi'(x)$ exists. By Corollary 1^0 of the preceding section, we have

$$\Psi(Y)=\int_Y (\nu)\Psi'(x)d\nu(x) \tag{3}$$

for bounded Borel sets Y whenever Ψ is an additive set-function absolutely continuous (ν). It follows that Φ, and hence W also, vanishes for every bounded Borel set contained in $A-B$. Consequently, decomposing $A-B$ into a disjoint sequence of bounded Borel sets, we get at once $\tilde{W}(A-B)=0$, and we derive similarly that $\tilde{\mu}(A-B)=0$. Since obviously $\tilde{\nu}(Z)=\tilde{\mu}(Z)+\tilde{W}(Z)$ for any set Z, we obtain

$$\tilde{\nu}(\boldsymbol{R}-B)=\tilde{\nu}(\boldsymbol{R}-A)+\tilde{\nu}(A-B)=0, \tag{4}$$

which proves part (i) of the theorem.

re (ii): In order to obtain formulas (1) and (2) we may suppose in view of (4) that $X\subset B$. We classify the points x of B into a triple of disjoint Borel sets P, Q, R according as respectively

a) $(\nu)\mu'(x)=0<(\nu)\Phi'(x)$, so that $(\mu)\Phi'(x)=+\infty$,

b) $(\nu)\mu'(x)=0>(\nu)\Phi'(x)$, so that $(\mu)\Phi'(x)=-\infty$,

c) $(\nu)\mu'(x)>0$, so that $(\mu)\Phi'(x)$ is finite.

We then have manifestly $P=BE_{+\infty}$ and $Q=BE_{-\infty}$, from which we find $XP=XE_{+\infty}$ and $XQ=XE_{-\infty}$. But $\Phi(X)=\Phi(XP)+\Phi(XQ)+\Phi(XR)$, and thus for the proof of (1) it is enough to show that the integral on the right of (1) equals $\Phi(XR)$.

Now, taking $\Psi=\mu$ in (3) and applying the theorem on change of measure in integration (see Saks, p. 37), we infer that

$$(5) \qquad \int_X \frac{d\Phi}{d\mu} d\mu = \int_X \frac{d\Phi}{d\mu} \frac{d\mu}{d\nu} d\nu = \int_{XR} \frac{d\Phi}{d\nu} d\nu,$$

with obvious meaning of the symbols $d\Phi/d\mu$, etc. Indeed we clearly have $(\mu)\Phi'(x)\cdot(\nu)\mu'(x)=(\nu)\Phi'(x)$ everywhere on R, while $(\nu)\mu'(x)=0$ for all points of $P\cup Q$. But the last integral in (5) must be equal to $\Phi(XR)$ on account of (3). This establishes formula (1).

It remains to deduce formula (2). But (2) is an immediate consequence of (1), since in view of conditions a) and b) we find by (3) that $\Phi(PY)\geqq 0$ and $\Phi(QY)\leqq 0$ for bounded Borel sets Y. Part (ii) is thus established.

re (iii): The proof for part (iii) proceeds in the same way as in Saks, p. 128, with the help of the Corollary 3^0 of §7.

9. Change of measure in integration. The decomposition theorem of the preceding section enables us to complete the theorem on change of measure in integration (*vide* Saks, p. 37), at least in the one-dimensional case, and the following theorem may essentially be regarded as a final result of its kind. The proof is immediate and may be left out.

THEOREM. *Given a nonnegative additive set-function ν besides μ, let E denote the Borel set of the points t at which $(\mu)\nu'(t)=+\infty$. Then*

$$(6) \qquad \int_X f(x)d\nu(x) = \int_X f(x)\frac{d\nu}{d\mu}(x)d\mu(x) + \int_{XE} f(x)d\nu(x),$$

provided X is a bounded Borel set and $f(x)$ is a B-measurable function, defined on $\boldsymbol{R}$ and possessing a definite integral (ν), finite or infinite, over the set X.

Suppose, further, that the function f is integrable (ν) over some bounded Borel set A. Then (6), restricted to Borel sets $X\subset A$, expresses the Lebesgue decomposition on A of the indefinite integral on the left, corresponding to the measure μ, the ν-integral on the right being the function of singularities (μ) of the left-hand side.

References

[1] Ka. Iseki: On the covering theorem of Vitali, Proc. Japan Acad., **36**, 539-542 (1960).

[2] S. Saks: Theory of the Integral, Warszawa-Lwów (1937), reprinted by Hafner Publishing Company.