2. On Convergences of Sequences of Continuous Functions

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We assume here that X is always a completely regular T_1-space and functions are all continuous and real-valued. It is well known that various types of convergences are defined for sequences of continuous functions. In this paper, we shall give the characterizations of spaces in which one of eight types of convergences, listed in § 1, implies some other types of convergences.

1. **Definitions.** We denote by $\{f_n\}\to f$ that $\{f_n\}$ converges to f *pointwisely* on X. We denote $\{f_n\}\to f$ (*unif.*) that $\{f_n\}$ converges *uniformly* to f. $\{f_n\}\to f$ (*compact*) means that $\{f_n\}$ is uniformly convergent to f on every compact subset of X. We shall say that $\{f_n\}$ *converges locally uniformly* to $f(\{f_n\}\to f$ (*loc. unif.*)) if for every $\varepsilon>0$ and every point x, there is an open neighborhood U of x and some integer $m>0$ such that $U\ni y$ implies that $|f_n(y)-f(y)|<\varepsilon$ for every $n>m$. $\{f_n\}$ is said to be *strictly continuously convergent* to $f(\{f_n\}\to f$ (*str. cont.*)) if $\{f(x_n)\}\to\alpha$ implies $\{f_n(x_n)\}\to\alpha$. $\{f_n\}$ is said to be *continuously convergent* to $f(\{f_n\}\to f$ (*cont.*)) if $\{x_n\}\to x$ implies $\{f_n(x_n)\}\to f(x)$. f is said to be *convergent to* f *quasi-uniformly* ($\{f_n\}\to f$ (*quasi-unif.*)) if $\{f_n\}\to f$ and if for every $\varepsilon>0$ and $m>0$, there exists a finite number of n_i, $n_i>m$ $(i=1,2,\cdots,k)$ such that for every x in X, $|f_{n_i}(x)-f(x)|<\varepsilon$ holds for at least one n_i. $\{f_n\}$ is said to be *almost uniformly convergent* to $f(\{f_n\}\to f$ (*almost unif.*)) if every subsequence of $\{f_n\}$ is quasi-uniformly convergent to f.

In the following, in case one type of convergences implies always the other one type of convergences, for instance, $\{f_n\}\to f$ (*unif.*) implies always $\{f_n\}\to f$, then we write [*unif.*]→[*pointwise*]. If U is an open set, then a non-negative continuous function f such that $f=0$ on $X-U$, $0\le f\le 1$ and $f(x)=1$ for some x in U is said to be an *associated function with* U. For a given U containing two points at least, there are many associated functions. If f is an associated function with an open set U, then $V=\{x; f(x)>1/2\}$ is said to be an *f-section* of U. Let $\{U_n\}$ be a family of open sets and let f_n be an associated function with U_n; then $\{f_n\}$ is said to be a *sequence associated with* $\{U_n\}$.

2. **Constructions of sequences of functions.** (2.1) *Suppose that X is not finite.* Then there is a sequence $\{x_n\}$ such that each point x_n is isolated in $\{x_n\}$. Thus there is, by the regularity, a family $\{U_n\}$

of open sets such that $\bar{U}_n \frown \bar{U}_m = \theta$ (an empty set). In the sequel, such a family $\{U_n\}$ is said to be *a family of closure-disjoint open sets.* Let $\{f_n\}$ be a sequence associated with $\{U_n\}$. It is obvious that $\{f_n\} \to \varphi$ (*almost unif.*) but $\{f_n\} \not\to \varphi$ (*str. cont.*) where φ denotes a function such that $\varphi(x)=0$ on X.

(2.2) *Suppose that X has a convergent sequence $\{x_n\} \to x$ where $x_n \neq x_m$ $(m \neq n)$ and $x_n \neq x$ for all n.* Let $\{U_n\}$ be a family of closure-disjoint open sets such that $U_n \ni x_n$ and let $\{f_n\}$ be a sequence associated with $\{U_n\}$ such that $f_n(x_n)=1$ for every n. It is easy to see that $\{f_n\} \to \varphi$ (*almost unif.*) but $\{f_n\} \not\to \varphi$ (*cont.*).

(2.3) *Suppose that X is not pseudo-compact.* Let $\{U_n\}$ be a locally finite family of closure-disjoint open sets and let $\{f_n\}$ be a sequence associated with $\{U_n\}$.

(α) *$\{f_n\} \to \varphi$ (loc. unif.), $\{f_n\} \to \varphi$ (almost unif.) but $\{f_n\} \not\to \varphi$ (str. cont.).* Since $\{U_n\}$ is locally finite, for any point x, there is an open neighborhood which intersects finitely many U_n. Thus $\{f_n\} \to \varphi$ (*loc. unif.*). The other parts are obvious.

Let us put $g_n(x) = \sum_{m \geq n} f_m(x)$ and $k_m(x) = \min\left(\sum_{n=1}^{\infty} n f_n(x), m\right)$.

(β) *$\{g_n\} \to \varphi$ (loc. unif.) but $\{g_n\} \not\to \varphi$ (quasi-unif.).* The first part follows from the facts that for a given point x, there are an open set U containing x and $m>0$ such that $g_n(y)=g_k(y)$ for $n, k>m$ and $y \in U$. The latter part is obvious.

(γ) *$\{k_n\} \to F = \sum_{n=1}^{\infty} n f_n$ (str. cont.) but $\{k_n\} \not\to F$ (quasi-unif.).* Since $\{U_n\}$ is locally finite, $F(x)$ is continuous. Suppose that $\{F(x_n)\} \to \alpha$. For any $\varepsilon>0$, there is m such that $m>\alpha$ and $|F(x_n)-\alpha|<\varepsilon$ for all $n>m$. This means that $F(x_n)=k_{m+1}(x_n)$ for $n>m$. Thus we have $k_n(x_n) \to \alpha$. The latter part is obvious.

Let us put $h_{2n}=f_{2n}$ and $h_{2n-1}=g_{2n-1}$ $(n=1, 2, \cdots)$.

(δ) *$\{h_n\} \to \varphi$ (loc. unif.), $\{h_n\} \to \varphi$ (quasi-unif.) but $\{h_n\} \not\to \varphi$ (str. cont.) and $\{h_n\} \not\to \varphi$ (almost unif.).* Since $\{h_{2n-1}\} \not\to \varphi$ (*quasi-unif.*), we have $\{h_n\} \not\to \varphi$ (*almost unif.*). The other parts are obvious.

3. Next implications are obvious: [*unif.*]$\to$[*almost unif.*]$\to$[*quasi-unif.*], [*unif.*]$\to$[*str. cont.*]$\to$[*cont.*]$\to$[*pointwise*] and [*compact*]$\to$[*cont.*].

Theorem 1. [*str. cont.*]$\to$[*loc. unif.*]$\to$[*compact*].

Proof. i) If $\{f_n\} \to f$ (*str. cont.*) but $\{f_n\} \not\to f$ (*loc. unif.*), then there is a point x such that $\{f_n\} \not\to f$ (*unif.*) on some neighborhood U of x, and hence some $\varepsilon>0$, there is a sequence $\{x_n\}$[1] in U such that $|f_n(x_n)-f(x_n)| \geq \varepsilon$. We lose no generality by assuming that $U \ni y$

1) There is, in details, a sequence $\{x_n\}$ containing a subsequence $\{x_{n_i}\}$ such that $|f_{n_i}(x_{n_i})-f(x_{n_i})| \geq \varepsilon$. But we shall write simply $\{x_n\}$ in stead of $\{x_{n_i}\}$ where no ambiguity arises from the abbreviation.

implies $|f(x)-f(y)|<\varepsilon$. Thus, since $\{f(x_n)\}$ is bounded in a real line, $\{f(x_n)\}$ has a cluster point α. Let us put $\{f(x_{n_i})\}\to\alpha$. By the assumption, we have $\{f_{n_i}(x_{n_i})\}\to\alpha$. This is a contradiction.

ii) Let $\{f_n\}\to f$ (*loc. unif.*) and F be a compact subset of X. For each x in F, there is an open neighborhood $U(x)$ on which $\{f_n\}\to f$ (*unif.*), that is, for a given $\varepsilon>0$, there is an integer $m(x)>0$ such that $|f_n(y)-f(y)|<\varepsilon$ for all $n>m(x)$ and all $y\in U(x)$. Since F is compact, we select a finite open subcovering $\{U(x_i);\ i=1,2,\cdots,n\}$ of $\{U(x);\ x\in F\}$. Let $m=\max\{m(x_1),\cdots,m(x_n)\}$; it is obvious that $|f(y)-f_n(y)|<\varepsilon$ for all $n>m$ and all $y\in F$.

Theorem 2. *The following conditions are equivalent:* 1) *X is pseudo-compact,* 2) *[pointwise]→[almost unif.],* 3) *[str. cont.]→[quasi-unif.],* 4) *[str. cont.]→[unif.],* 5) *[loc. unif.]→[unif.] and* 6) *[loc. unif.]→[str. cont.].*

Proof. 1)→2) (from [1])→3)→1) (by (2.3, γ)). 1)↔4) [3, Theorem 2]↔5) [2, Theorem 2]→6)→1) (by (2.3, α)).

Theorem 3. *The following conditions are equivalent:* 1) *X has no convergent sequences,* 2) *[pointwise]→[cont.],* 3) *[quasi-unif.]→[cont.] and* 4) *[almost unif.]→[cont.].*

Proof. If there are no convergent sequences, $\{f_n\}\to f$ implies always $\{f_n\}\to f$ (*cont.*). Thus we have 1)→2). 2)→3)→4) are obvious. 4)→1) follows from (2.2).

Theorem 4. *The following conditions are equivalent:* 1) *any compact subset of X is finite,* 2) *[pointwise]→[compact],* 3) *[quasi-unif.]→[compact] and* 4) *[almost unif.]→[compact].*

Proof. 1)→2)→3)→4) are obvious. Now suppose that there is a compact subset F which is not finite. Let $\{x_n\}$ be a sequence in F such that each x_n is isolated in $\{x_n\}$ and let $\{U_n;\ U_n\ni x_n\}$ be a family of closure-disjoint open sets and $\{f_n\}$ be a sequence associated with $\{U_n\}$. It is easy to see that $\{f_n\}\to\varphi$ (*almost unif.*) but $\{f_n\}\not\to\varphi$ (*compact*).

Theorem 5.[2] *The following conditions are equivalent:* 1) *for any family $\{U_n\}$ of closure-disjoint open sets, there is a sequence $\{y_n;\ y_n\in U_n\}$ which contains a convergent subsequence,* 2) *[cont.]→[unif.] and* 3) *[cont.]→[str. cont.].*

Proof. 2)→3) is obvious. We shall prove 1)→2). Let $\{f_n\}\to f$ (*cont.*) and $\{f_n\}\not\to f$ (*unif.*); then there is, for some $\varepsilon>0$, a sequence $\{x_n\}$[1] such that $|f_n(x_n)-f(x_n)|\geq\varepsilon$. We lose no generality by assuming that $x_n\neq x_m$ $(n\neq m)$ and each x_n is isolated in $\{x_n\}$. Thus there is a family $\{U_n;\ U_n\ni x_n\}$ of closure-disjoint open sets where $U_n\ni y$ implies

2) $(a)\to(b)$ in Theorem 1 in [4] is in the wrong. In a Čech-compactification of a discrete space, $(a)\to(b)$ does not hold (see Theorem 3 in [5]—a function g_n defined in iii) should be read as follows: $g_n(a_n)=1$ for all n and $g_n(z)=0$ for $z\neq a_n$).

that $|f_n(x_n)-f_n(y)|<\varepsilon/2^n$ and $|f(x_n)-f(y)|<\varepsilon/2^n$. By the assumption, there is a convergent subsequence $\{y_{n_i}\}(\to y)$. Since $\{f_n\}\to f$ (*cont.*), $\{f(y_{n_i})\}\to f(y)$ implies $f_{n_i}(y_{n_i})\to f(y)$. On the other hand, since $|f_{n_i}(x_{n_i})-f_{n_i}(y_{n_i})|<\varepsilon/2^n$ and $|f(x_{n_i})-f(y_{n_i})|<\varepsilon/2^n$, we have $\lim_i f_{n_i}(x_i)=\lim_i f(x_{n_i})$. This is a contradiction.

By (2.3, α), 2) or 3) implies the pseudo-compactness of X. Thus, to prove 3)$\to$1), it is sufficient to show, by Theorem 2, 2)$\to$1). Suppose that there is a family $\{U_n\}$ of closure-disjoint open sets in which there are no sequences having convergent subsequences. Let $\{f_n\}$ be a sequence associated with $\{U_n\}$. Let $\{z_n\}\to z$. By the assumption on $\{U_n\}$, $\bigcup_{n=1}^{\infty} U_n$ contains at most finitely many points z_n. Thus for sufficiently large n, we have $f_n(z_n)=0$, in other words, $f_n(z_n)\to\varphi(z)=0$. Since f_n is associated with U_n, there is a point $x_n \in U_n$ such that $f_n(x_n)=1$, thus it is obvious that $\{f_n\}\not\to\varphi$ (*unif.*).

Theorem 6. *The following conditions are equivalent:* 1) *for any family $\{U_n\}$ of closure-disjoint open sets, there is a sequence $\{x_n; x_n \in U_n\}$ which contains a subsequence contained in a compact set,* 2) *[compact]$\to$[unif.] and* 3) *[compact]$\to$[str. cont.].*

Proof. 2)$\to$3) is obvious. We shall show 1)$\to$2). If $\{f_n\}\to f$ (*compact*) and $\{f_n\}\not\to f$ (*unif.*), there is, for some $\varepsilon>0$, a family $\{U_n\}$ of closure-disjoint open sets such that $\{U_n\}\ni x_n$,[1] $|f_n(x_n)-f(x_n)|\ge\varepsilon$ and $U_n\ni y$ implies that $|f_n(x_n)-f_n(y)|<\varepsilon/2^n$ and $|f(x_n)-f(y)|<\varepsilon/2$. Let $\{y_{n_i}; y_{n_i}\in U_{n_i}\}$ be a sequence contained in some compact set F. Since $\{f_n\}\to f$ (*unif.*) on F, we have $|f_n(y)-f(y)|<\varepsilon/4$ for $y\in F$ and for all $n>m$ where $m(>0)$ is some fixed integer. From the following inequalities: $|f_{n_i}(y_{n_i})-f(y_{n_i})|<\varepsilon/4$, $|f_{n_i}(x_{n_i})-f_{n_i}(y_{n_i})|<\varepsilon/2^n$ and $|f(x_{n_i})-f(y_{n_i})|<\varepsilon/2^n$, we have $|f(x_{n_i})-f_{n_i}(x_{n_i})|<\varepsilon$. This is a contradiction.

Conversely, by Theorem 1 and (2.3, δ), 2) or 3) implies the pseudo-compactness of X. Thus, to prove 3)$\to$1), it is sufficient, by Theorem 2, to show that 2)$\to$1). Suppose that there is a family $\{U_n\}$ of closure-disjoint open sets which has no sequences having subsequences contained in a compact subset. Let $\{f_n\}$ be a sequence associated with $\{U_n\}$. Then, since any compact subset intersects only finitely many U_n, we have $\{f_n\}\to\varphi$ (*compact*). On the other hand, it is easy to see that $\{f_n\}\not\to\varphi$ (*unif.*).

Theorem 7. *The following conditions are equivalent:* 1) *X is finite,* 2) *[pointwise]$\to$[unif.] and* 3) *[almost unif.]$\to$[str. cont.].*

Proof. 1)$\to$2)$\to$3) are obvious. 3)$\to$1) follows from (2.1).

Theorem 8. *[cont.]$\to$[compact] is equivalent to the property that if each U_n of family $\{U_n\}$ of closure-disjoint open sets intersects with a fixed compact set, then there is a sequence $\{x_n; x_n \in U_n\}$ which* *contains a convergent subsequence.*

Proof. If $\{f_n\}\to f$ (*cont.*) but $\{f_n\}\not\to f$ (*compact*), there is a compact set F on which $\{f_n\}\not\to f$ (*unif.*). Therefore there is, for some $\varepsilon>0$, a point x_n,[1] for each n, in F such that $|f_n(x_n)-f(x_n)|>\varepsilon$. We can assume that x_n is isolated in $\{x_n\}$. Let $\{U_n; x_n \in U_n\}$ be a family of closure-disjoint open sets. By the assumption, there is a sequence $\{y_n; y_n \in U_n\}$ which contains a convergent subsequence $\{y_{n_i}\}(\to y)$. Since $\{f_n\}\to f$ (*cont.*) and f is continuous, we have $\{f_{n_i}(y_{n_i})\}\to f(y)$ and $\{f(y_{n_i})\}\to f(y)$. This is a contradiction.

Conversely suppose that there is a family $\{U_n\}$ of closure-disjoint open sets such that each U_n intersects with a fixed compact set F and any sequence $\{x_n; x_n \in U_n\}$ has no convergent subsequences. Let $\{f_n\}$ be a sequence associated with $\{U_n\}$; then it is easy to see that $\{f_n\}\to\varphi$ (*cont.*) but $\{f_n\}\not\to\varphi$ (*compact*).

Theorem 9. *[compact]$\to$[loc. unif.] is equivalent to the property that for any point-finite family $\{U_n\}$ of open sets and for a sequence $\{f_n\}$ of associated functions with $\{U_n\}$ such that $\overline{\left(\bigcup_{n=1}^{\infty} V_n\right)}-\bigcup_{n=1}^{\infty}\overline{V}_n\neq\theta$ where V_n is an f_n-section of U_n, there is a compact set intersecting infinitely many U_n.*

Proof. Suppose that there is a point-finite family $\{U_n\}$ of open sets and a sequence $\{f_n\}$ of associated functions with $\{U_n\}$ such that $M=\overline{\left(\bigcup_{n=1}^{\infty} V_n\right)}-\bigcup_{n=1}^{\infty}\overline{V}_n\neq\theta$ where V_n is an f_n-section of U_n, but there is no compact subset intersecting infinitely many U_n. Since $\{U_n\}$ is point-finite, it is easy to see $\{f_n\}\to\varphi$ on X. Moreover, but the assumption, any compact set intersects only finitely many U_n, and hence $\{f_n\}\to\varphi$ (*compact*). On the other hand, let $x\in M$. Any open neighborhood W of x contains x_{n_i} in V_{n_i} $(i=1, 2,\cdots)$. Thus $\{f_{n_i}\}\not\to\varphi$ (*unif.*) on W, that is, $\{f_n\}\not\to\varphi$ (*loc. unif.*).

Conversely let $\{f_n\}\to f$ (*compact*) but $\{f_n\}\not\to f$ (*loc. unif.*). Then there are some $\varepsilon>0$ and a point x such that $\{f_n\}\not\to f$ (*unif.*) on every neighborhood of x. Thus, since $\{f_n\}\to f$, $\overline{\left(\bigcup_{n=1}^{\infty} O(n;\varepsilon)\right)}-\bigcup_{n=1}^{\infty}\overline{O(n;\varepsilon)}\ni x$ where $O(n;i)=\{y; |f_n(y)-f(y)|>i\}$ and we shall sum up n, in case of need, greater than some fixed integer $n_0>0$. If we put

$$g_n(y)=\begin{cases}(f_n(y)-f(y))/\varepsilon-1/2 & \text{for } y; |f_n(y)-f(y)|>\varepsilon/2,\\ 0 & \text{for } y; |f_n(y)-f(y)|\leq\varepsilon/2.\end{cases}$$

Then g_n is an associated function with $O(n;\varepsilon/2)$ and $O(n;\varepsilon)=\{y; g(y)>1/2\}$ is a g_n-section of $O(n;\varepsilon/2)$. It is an immediate consequence of $\{f_n\}\to f$ that $\{O(n;\varepsilon/2)\}$ is point-finite. Thus by the assumption, there is a compact set F intersecting infinitely many $O(n;\varepsilon/2)$. This shows that $\{f_n\}\not\to f$ (*unif.*) on F, and hence $\{f_n\}\not\to f$ (*compact*). This is a contradiction.

References

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