25. *On the Unitary Equivalence of Normal Operators in Hilbert Spaces*

By Sakuji INOUE
Faculty of Education, Kumamoto University
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The purpose of this paper is to find a necessary and sufficient condition for the unitary equivalence of normal operators in the abstract Hilbert space $\mathfrak{H}$ which is complete, separable, and infinite-dimensional.

Definition. If we denote by $\mathfrak{M}^{\alpha}$ the eigenspace determined by all eigenelements of a normal operator N in $\mathfrak{H}$ corresponding to the eigenvalue α, the projection operator of $\mathfrak{H}$ on $\mathfrak{M}^{\alpha}$ is called the eigenprojector corresponding to the eigenvalue α of N.

Theorem 1. Let N_1 and N_2 be normal operators in $\mathfrak{H}$ such that the sum of all eigenprojectors of N_j is identical with the identity operator I for each value of $j=1, 2$. Then for the unitary equivalence of N_1 and N_2 it is necessary and sufficient that N_1 and N_2 have the same continuous spectrum and same point spectrum (inclusive of the multiplicities of eigenvalues).

Proof. From the fact that the spectral classification of the points on the complex plane for N_j (inclusive of the multiplicities of eigenvalues) is invariant under the unitary transformation UN_jU^{-1} for any unitary operator U, it is clear that the condition given in the theorem is necessary; hence it remains only to prove the sufficiency of the condition.

Let $\{\varphi_\nu^{(j)}\}$ be an orthonormal set of all eigenelements of N_j; let $\{l_n\}$ and Δ be the common point spectrum and common continuous spectrum of N_1 and N_2 respectively; and let $\{P_j(z)\}$, $\{E_j(\lambda)\}$ and $\{F_j(\mu)\}$ be the spectral families of N_j, the self-adjoint operators $H_j=\frac{1}{2}(N_j+N_j^*)$ and $K_j=\frac{1}{2i}(N_j-N_j^*)$ respectively. Then, by hypotheses, $\{\varphi_\nu^{(1)}\}$ and $\{\varphi_\nu^{(2)}\}$ are complete orthonormal sets respectively and can be put in one-to-one correspondence in such a way that corresponding elements are eigenelements for N_1 and N_2 respectively, corresponding to the same eigenvalue; and in addition, since the residual spectrum of N_j is empty and since the spectral representation of N_j vanishes on the resolvent set,

$$N_j=\sum_n l_n P_j^{(n)}+\int_\Delta z dP_j(z), \quad H_j=\sum_n \Re(l_n)P_j^{(n)}+\int_\Delta \Re(z)dP_j(z),$$

$$K_j=\sum_n \mathfrak{I}(l_n)P_j^{(n)}+\int_\Delta \mathfrak{I}(z)dP_j(z),$$

where $P_j^{(n)}$ denotes the eigenprojector of N_j corresponding to the eigenvalue l_n, each of the three projection-integrals vanishes by virtue of the hypothesis $\sum_n P_j^{(n)}=I$, and

$$\begin{aligned} P_j^{(n)}&=[E_j(\mathfrak{R}(l_n))-E_j(\mathfrak{R}(l_n)-0)][F_j(\mathfrak{I}(l_n))-F_j(\mathfrak{I}(l_n)-0)] \\ &=E_j(\mathfrak{R}(l_n))-E_j(\mathfrak{R}(l_n)-0)=F_j(\mathfrak{I}(l_n))-F_j(\mathfrak{I}(l_n)-0), \\ &\qquad (j=1,2;\ n=1,2,\cdots), \end{aligned} \tag{1}$$

as can be easily verified from the respective spectral representations of N_j, H_j and K_j with respect to $\{P_j(z)\}$, $\{E_j(\lambda)\}$ and $\{F_j(\mu)\}$. It follows therefore that the point spectra of H_j and K_j are given by $\{\mathfrak{R}(l_n)\}$ and $\{\mathfrak{I}(l_n)\}$ respectively and that $\{\varphi_\nu^{(j)}\}$ is not only an orthonormal set of all eigenelements of H_j but also that of K_j. In consequence, according to a well-known theorem concerning the unitary equivalence of self-adjoint operators, there exist unitary operators U and V such that the equalities $H_1=UH_2U^{-1}$ and $K_1=VK_2V^{-1}$ hold. These results permit us to assert that

$$\begin{aligned} \int_G \mathfrak{R}(z)dP_1(z)&=\int_G \mathfrak{R}(z)d[UP_2(z)U^{-1}] \\ &=\sum_n \mathfrak{R}(l_n)U[E_2(\mathfrak{R}(l_n))-E_2(\mathfrak{R}(l_n)-0)][F_2(\mathfrak{I}(l_n))-F_2(\mathfrak{I}(l_n)-0)]U^{-1}, \end{aligned}$$

where G denotes the complex z-plane, and that similarly

$$\begin{aligned} &\int_G \mathfrak{I}(z)dP_1(z) \\ &=\sum_n \mathfrak{I}(l_n)V[E_2(\mathfrak{R}(l_n))-E_2(\mathfrak{R}(l_n)-0)][F_2(\mathfrak{I}(l_n))-F_2(\mathfrak{I}(l_n)-0)]V^{-1}. \end{aligned}$$

On the other hand, the equality $H_1=UH_2U^{-1}$ implies that $E_1(\lambda)=UE_2(\lambda)U^{-1}$, $-\infty<\lambda<\infty$, and hence that

$$E_1(\mathfrak{R}(l_n))-E_1(\mathfrak{R}(l_n)-0)=U[E_2(\mathfrak{R}(l_n))-E_2(\mathfrak{R}(l_n)-0)]U^{-1}, \tag{2}$$

and similarly the equality $K_1=VK_2V^{-1}$ implies that

$$F_1(\mathfrak{I}(l_n))-F_1(\mathfrak{I}(l_n)-0)=V[F_2(\mathfrak{I}(l_n))-F_2(\mathfrak{I}(l_n)-0)]V^{-1}. \tag{3}$$

Moreover, by applying (1) to (2) and (3) we obtain

$$\begin{aligned} &U[E_2(\mathfrak{R}(l_n))-E_2(\mathfrak{R}(l_n)-0)][F_2(\mathfrak{I}(l_n))-F_2(\mathfrak{I}(l_n)-0)]U^{-1} \\ &=V[E_2(\mathfrak{R}(l_n)-E_2(\mathfrak{R}(l_n)-0)][F_2(\mathfrak{I}(l_n))-F_2(\mathfrak{I}(l_n)-0)]V^{-1},\quad n=1,2,\cdots . \end{aligned}$$

These results established above lead us to the conclusion that

$$\begin{aligned} N_1&=\int_G \mathfrak{R}(z)dP_1(z)+i\int_G \mathfrak{I}(z)dP_1(z) \\ &=\sum_n l_nU[E_2(\mathfrak{R}(l_n))-E_2(\mathfrak{R}(l_n)-0)][F_2(\mathfrak{I}(l_n))-F_2(\mathfrak{I}(l_n)-0)]U^{-1} \\ &=\int_G zd[UP_2(z)U^{-1}] \\ &=UN_2U^{-1}. \end{aligned}$$

The given condition is therefore sufficient.

Corollary 1. If N_1 and N_2 are compact normal operators in $\mathfrak{H}$, then for the unitary equivalence of N_1 and N_2 it is necessary and sufficient that N_1 and N_2 have the same continuous spectrum and same point spectrum (inclusive of the multiplicities of eigenvalues).

Proof. Since, by hypothesis, an orthonormal set of all eigenelements of N_j is complete in $\mathfrak{H}$ for each value of $j=1, 2$, the present corollary is a direct consequence of Theorem 1.

Corollary 2. Let N_1 and N_2 be non-compact normal operators in $\mathfrak{H}$. If there exist a non-zero complex number α, positive integers p_j, and complete orthonormal sets $\{\psi_\nu^{(j)}\}$, $j=1, 2$, such that

$$\sum_{\nu=1}^{\infty} \|(N_j-\alpha I)^{p_j}\psi_\nu^{(j)}\|^2<\infty, \qquad j=1, 2,$$

then the same assertion as that stated in the preceding corollary holds.

Proof. Since, by hypotheses, it is verified without difficulty that $N_j-\alpha I$ is a compact normal operator for each value of $j=1, 2$, the present corollary follows at once from Corollary 1.

Theorem 2. Let N_1 and N_2 be normal operators in $\mathfrak{H}$ such that the sum of all eigenprojectors of N_j is less than the identity operator I for each value of $j=1, 2$; let $\{\varphi_\nu^{(2)}\}$ be an orthonormal set of all eigenelements of N_2; let $\{E_j(\lambda)\}$ and $\{F_j(\mu)\}$ be the spectral families of $H_j=\frac{1}{2}(N_j+N_j^*)$ and $K_j=\frac{1}{2i}(N_j-N_j^*)$, $j=1, 2$, respectively. Then for the unitary equivalence of N_1 and N_2 it is necessary and sufficient that

(i) N_1 and N_2 have the same continuous spectrum and same point spectrum (inclusive of the multiplicities of eigenvalues);

(ii) there exists a unitary operator U such that, for each value of $\nu=1, 2, \cdots$, $U\varphi_\nu^{(2)}$ is an eigenelement of N_1 for the eigenvalue of N_2 corresponding to $\varphi_\nu^{(2)}$;

(iii) for any element f belonging to the orthogonal complement $\mathfrak{N}_2$ of the subspace $\mathfrak{M}_2$ determined by $\{\varphi_\nu^{(2)}\}$ the relations

$$\|E_2(\lambda)f\|=\|E_1(\lambda)Uf\|, \tag{4}$$

$$\|F_2(\mu)f\|=\|F_1(\mu)Uf\| \tag{5}$$

hold on the common continuous spectrum of H_1 and H_2 and on that of K_1 and K_2 respectively.

Furthermore $N_2=U^{-1}N_1U$ for such a U as above.

Proof. Suppose that there exists a unitary operator U satisfying the condition $N_2=U^{-1}N_1U$. Then it is first clear that (i) is satisfied; and moreover there is no difficulty in showing that (ii) holds. If we now use the symbols $\{P_j(z)\}$, $j=1, 2$, and G defined before, then from the spectral representations of N_2 and $U^{-1}N_1U$ and from the uniqueness of the spectral family associated with a normal operator, we find at once that $P_2(z)=U^{-1}P_1(z)U$ on G and hence that

$$U^{-1}H_1U=\int_G \Re(z)d[U^{-1}P_1(z)U]=\int_G \Re(z)dP_2(z)=H_2.$$

The last relation implies that $E_2(\lambda)=U^{-1}E_1(\lambda)U$ on the interval $(-\infty, \infty)$. In an entirely similar manner, we see that $F_2(\mu)=U^{-1}F_1(\mu)U$ on $(-\infty, \infty)$. Hence (4) and (5) both hold on $(-\infty, \infty)$. In addition, it is evident that H_1 and H_2 have the same continuous spectrum and that the same is true of K_1 and K_2. The condition given in the statement of the present theorem is thus necessary for the unitary equivalence of N_1 and N_2.

Conversely we shall now suppose that the chain of conditions (i), (ii), and (iii) is satisfied.

If we denote by $\{l_n\}$ the common point spectrum of N_j, $j=1,2$, as before and if, by (ii), we suppose that $U\varphi_\nu^{(2)}$ and $\varphi_\nu^{(2)}$ are eigenelements for N_1 and N_2 respectively, corresponding to an arbitrarily given eigenvalue $l_n\in\{l_n\}$, then we see readily that $N_2\varphi_\nu^{(2)}=U^{-1}N_1U\varphi_\nu^{(2)}$. This relation yields the result that

(6) $$N_2=U^{-1}N_1U \quad \text{on } \mathfrak{M}_2\cap\mathfrak{D}(N_2),$$

where $\mathfrak{D}(N_2)$ denotes the domain of N_2.

We shall prove below that $N_2=U^{-1}N_1U$ on $\mathfrak{N}_2\cap\mathfrak{D}(N_2)$.

Since, by (i), clearly H_1 and H_2 have the same continuous spectrum, we denote it by $\Delta(H)$ and express symbolically by $x>\Delta(H)$ (or by $\Delta(H)<x$) the relation between $\Delta(H)$ and an arbitrary point x on $(-\infty, \infty)$ such that $\xi<x$ for every $\xi\in\Delta(H)$. Then from the fact that $E_2(\Re(l_n))-E_2(\Re(l_n)-0)$ is the eigenprojector of H_2 corresponding to the eigenvalue $\Re(l_n)$ it follows that, for $x>\Delta(H)$,

$$\begin{aligned}||E_2(x)f||^2&=\int_{-\infty}^{x} d||E_2(\lambda)f||^2 \qquad (f\in\mathfrak{N}_2)\\ &=\sum_{\Re(l_n)\leq x}||[E_2(\Re(l_n))-E_2(\Re(l_n)-0)]f||^2+\int_{\Delta(H)} d||E_2(\lambda)f||^2\\ &=\int_{\Delta(H)} d||E_2(\lambda)f||^2,\end{aligned}$$

where $\sum_{\Re(l_n)\leq x}$ denotes the sum for all eigenvalues $\Re(l_n)$ of H_2 such that $\Re(l_n)\leq x$, while

(7) $$\begin{aligned}||f||^2&=\int_{-\infty}^{\infty} d||E_2(\lambda)f||^2 \qquad (f\in\mathfrak{N}_2)\\ &=\sum_{\Re(l_n)}||[E_2(\Re(l_n))-E_2(\Re(l_n)-0)]f||^2+\int_{\Delta(H)} d||E_2(\lambda)f||^2\\ &=\int_{\Delta(H)} d||E_2(\lambda)f||^2,\end{aligned}$$

where $\sum_{\Re(l_n)}$ denotes the sum for all $\Re(l_n)$. As a result, we obtain $||E_2(x)f||^2=||f||^2$ for every $x>\Delta(H)$ and for any $f\in\mathfrak{N}_2$. On the other hand, as can be found immediately from the above reasoning, the relation $\|E_2(x)f\|^2=0$ holds for every $x<\Delta(H)$ and for any $f\in\mathfrak{N}_2$. We next consider a point $x\in(-\infty,\infty)$ such that $\Delta'<x<\Delta''$ or $\Delta'\leqq x<\Delta''$ where $\Delta'\cup\Delta''=\Delta(H)$ and $\Delta'\leqq x$ denotes that $(\Delta'-x)<x\in\Delta'$. Then, for the points x,ζ with $\Delta'\leqq\zeta<\Delta''$ and for $f\in\mathfrak{N}_2$, we have

$$\|E_2(x)f\|^2=\int_{-\infty}^{x}d\|E_2(\lambda)f\|^2=\int_{-\infty}^{\zeta}d\|E_2(\lambda)f\|^2=\|E_2(\zeta)f\|^2.$$

In addition, by making use of (4) and (7) we have

$$\begin{aligned}\|f\|^2&=\int_{-\infty}^{\infty}d\|E_1(\lambda)Uf\|^2 \qquad (f\in\mathfrak{N}_2)\\ &=\sum_{\mathfrak{R}(l_n)}\|[E_1(\mathfrak{R}(l_n))-E_1(\mathfrak{R}(l_n)-0)]Uf\|^2+\int_{\Delta(H)}d\|E_1(\lambda)Uf\|^2\\ &=\sum_{\mathfrak{R}(l_n)}\|[E_1(\mathfrak{R}(l_n))-E_1(\mathfrak{R}(l_n)-0)]Uf\|^2+\|f\|^2,\end{aligned}$$

and hence $[E_1(\mathfrak{R}(l_n))-E_1(\mathfrak{R}(l_n)-0)]Uf=0, \quad (n=1,2,\cdots)$.

Since the final relations show that Uf, $(f\in\mathfrak{N}_2)$, belongs to the orthogonal complement $\mathfrak{N}_1$ of the subspace determined by all eigenelements of N_1, we can verify with the help of the same reasoning as above that

$$\|E_1(x)Uf\|^2=\begin{cases}\|f\|^2 & (x>\Delta(H)),\\ 0 & (x<\Delta(H)),\\ \|E_1(\zeta)Uf\|^2 & (\Delta'<x<\Delta'',\text{ or }\Delta'\leqq x<\Delta'';\ \Delta'\leqq\zeta<\Delta'').\end{cases}$$

Consequently the condition that the relation (4) holds on $\Delta(H)$ implies that it holds on $(-\infty,\infty)$.

In an entirely similar manner we can find that, if the relation (5) holds on the common continuous spectrum of K_1 and K_2, it holds on $(-\infty,\infty)$. Accordingly the relations

$$\left.\begin{aligned}&(8)\qquad ((E_2(\lambda)-U^{-1}E_1(\lambda)U)f,f)=0\\ &(9)\qquad ((F_2(\mu)-U^{-1}F_1(\mu)U)f,f)=0\end{aligned}\right\},\quad (f\in\mathfrak{N}_2),$$

are valid for every $\lambda,u\in(-\infty,\infty)$. Moreover we can prove as below that both $(E_2(\lambda)-U^{-1}E_1(\lambda)U)f$ and $(F_2(\mu)-U^{-1}F_1(\mu)U)f$, where $\lambda,u\in(-\infty,\infty)$ and $f\in\mathfrak{N}_2$, belong to $\mathfrak{N}_2$, and then that both $E_2(\lambda)=U^{-1}E_1(\lambda)U$ and $F_2(\mu)=U^{-1}F_1(\mu)U$ hold on $\mathfrak{N}_2\cap\mathfrak{D}(N_2)$.

In the first place, by virtue of the fact that $\int_{\Delta(H)}dE_2(\lambda)$ is the projector of $\mathfrak{H}$ on $\mathfrak{N}_2$ we have for every $x>\Delta(H)$

$$E_2(x)f=\int_{-\infty}^{x}dE_2(\lambda)f=\int_{\Delta(H)}dE_2(\lambda)f=f \qquad (f\in\mathfrak{N}_2),$$

and for every $x<\Delta(H)$

$$E_2(x)f=\int_{-\infty}^{x}dE_2(\lambda)f=0 \qquad (f\in\mathfrak{N}_2).$$

We next consider such a point x with $\Delta'<x<\Delta''$ (or with $\Delta'\leqq x<\Delta''$) as described before. Then

$$E_2(x)f=\int_{-\infty}^{\zeta} dE_2(\lambda)f=\int_{\Delta'} dE_2(\lambda)f \qquad (f\in\mathfrak{N}_2),$$

where $\Delta'\leqq\zeta<\Delta''$. Since, in addition, $\int_{\Delta'} dE_2(\lambda)$ is a projector permutable with $\int_{\Delta(H)} dE_2(\lambda)$, and since $\int_{\Delta'} dE_2(\lambda)\cdot\int_{\Delta(H)} dE_2(\lambda)=\int_{\Delta'} dE_2(\lambda)$, $\int_{\Delta'} dE_2(\lambda)<\int_{\Delta(H)} dE_2(\lambda)$, that is, $\int_{\Delta'} dE_2(\lambda)\cdot\mathfrak{H}\subset\mathfrak{N}_2$. We find from these results that $E_2(\lambda)f$ belongs to $\mathfrak{N}_2$ for every $\lambda\in(-\infty,\infty)$ and for any $f\in\mathfrak{N}_2$.

Remembering that Uf, $(f\in\mathfrak{N}_2)$, belongs to $\mathfrak{N}_1$, we can easily find by similar reasoning that $U^{-1}E_1(\lambda)Uf$ belongs to $\mathfrak{N}_2$ for every $\lambda\in(-\infty,\infty)$ and for any $f\in\mathfrak{N}_2$. Thus we find that $(E_2(\lambda)-U^{-1}E_1(\lambda)U)f$ belongs to $\mathfrak{N}_2$ for every $\lambda\in(-\infty,\infty)$ and for any $f\in\mathfrak{N}_2$.

On the other hand, if for brevity of expression we denote by T the self-adjoint operator $E_2(\lambda)-U^{-1}E_1(\lambda)U$, the relation

$$(Tg,h)=\left\{\left(T\frac{g+h}{2},\ \frac{g+h}{2}\right)-\left(T\frac{g-h}{2},\ \frac{g-h}{2}\right)\right\}$$
$$+i\left\{\left(T\frac{g+ih}{2},\ \frac{g+ih}{2}\right)-\left(T\frac{g-ih}{2},\ \frac{g-ih}{2}\right)\right\}$$

holds, in general, for every pair of elements $g, h\in\mathfrak{H}$. Applying this relation to (8), we obtain the relation $E_2(\lambda)=U^{-1}E_1(\lambda)U$ holding on $\mathfrak{N}_2$ for every $\lambda\in(-\infty,\infty)$, because of the facts that $((E_2(\lambda)-U^{-1}E_1(\lambda)U)g,h)=0$, $-\infty<\lambda<\infty$, holds for every pair of $g,h\in\mathfrak{N}_2$ and $(E_2(\lambda)-U^{-1}\times E_1(\lambda)U)g$, $-\infty<\lambda<\infty$, belongs to $\mathfrak{N}_2$ for every $g\in\mathfrak{N}_2$. The final relation implies that $H_2=U^{-1}H_1U$ on $\mathfrak{N}_2\cap\mathfrak{D}(N_2)$.

Moreover, by reasoning entirely like that used to (8) we can establish the relation $K_2=U^{-1}K_1U$ holding on $\mathfrak{N}_2\cap\mathfrak{D}(N_2)$; and the last two relations imply that

(10) $$N_2=U^{-1}N_1U \quad \text{on } \mathfrak{N}_2\cap\mathfrak{D}(N_2).$$

Since $\mathfrak{H}=\mathfrak{M}_2\oplus\mathfrak{N}_2$, the relations (6) and (10) enable us to conclude that $N_2=U^{-1}N_1U$ on $\mathfrak{D}(N_2)$; hence the condition given in the present theorem is sufficient for the unitary equivalence of N_1 and N_2.

Remark. Combining (4) and (5), we have the relation

$$\|P_2(z)f\|=\|P_1(z)Uf\|$$

holding on the common continuous spectrum of N_1 and N_2 for every $f\in\mathfrak{N}_2$; and the case where the point spectra of N_j, $j=1,2$, are empty is trivial.