# 102. *Remarks on Cantor's Absolute. II*

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As to the notions and notations we refer to [1, 2] throughout this paper. In [2], the author presented Cantor's Absolute as a universe satisfying certain conditions. In addition to these conditions we shall now assume the following:

For any set $a$ and any well-defined univalent relation, there exists a set consisting of all sets each of which corresponds to an element of $a$ by the relation.

In assuming this, we shall prove in this paper that, for every definable class of true closed formulas in $T_C$, a formula with the following meaning belongs to $T_C$:

For every set $a$ of $C$, there exists a set $b$ which has $a$ as an element and which is a super-complete model of all formulas of the class under consideration.

More exactly, this assertion is given in the following form, if we use $\mathfrak{A}$ and $\mathfrak{D}$ in the same meaning as in [2]:

(*) $\forall u \exists x(u \in x \wedge \forall y \forall z(y \in x \wedge (z \subseteq y \vee z \in y) \mid\!\!\!- z \in x) \wedge \forall y(\mathfrak{A}(y) \mid\!\!\!- \mathfrak{D}(x, y)))$.

This is an extension of the problem (B) in [2], and we left (B) as an open problem.

First we shall define some concepts. Let us extend the notion of 'set theory' in [2] to a set theory in the first order predicate calculus, which consists of not only the predicate $\varepsilon$, logical symbols and bound variables, but finitely or infinitely many individual constants. If $T$ is such a set theory which contains $a_0, a_1, \cdots$ as indivisual constants, we call $T$ a set theory with $a_0, a_1, \cdots$.

Let $T$ be a set theory with $a_0, a_1, \cdots$ and $B_T$ be the class consisting of all $\{x\}\mathfrak{A}(x, a_0, a_1, \cdots)$, where $\{x\}\mathfrak{A}(x, a_0, a_1, \cdots)$ consists only of logical symbols, the predicate $\varepsilon$, bound variables and $a_0, a_1, \cdots$, and it will be abbreviated as $\{x\}\mathfrak{A}(x)$ if no confusion is to be feared. $T$ is called to be 'definite', if it satisfies the following conditions:

1) $T$ is complete.

2) If $\exists x \mathfrak{A}(x)$ belongs to $T$, then there exists a formula $\exists x \mathfrak{B}(x)$ such that $\exists x \mathfrak{B}(x)$, $\forall x \forall y(\mathfrak{B}(x) \wedge \mathfrak{B}(y) \mid\!\!\!- x = y)$ and $\exists x(\mathfrak{A}(x) \wedge \mathfrak{B}(x))$ belong to $T$. Let $\{x\}\mathfrak{A}(x)$ and $\{x\}\mathfrak{B}(x)$ belong to $B_T$. We define '$\{x\}\mathfrak{A}(x)$ belongs to the same class as $\{x\}\mathfrak{B}(x)$ with respect to $T$', if and only if $\forall x(\mathfrak{A}(x) \mid\!\!\!-\!\!\!\mid \mathfrak{B}(x))$ belongs to $T$. The class which contains $\{x\}\mathfrak{A}(x)$ is written $(\{x\}\mathfrak{A}(x))$ and $\{x\}\mathfrak{A}(x)$ is said to represent the class. A class

$(\{x\}\mathfrak{A}(x))$ is said to be *definite with respect to T*, if $\exists x\mathfrak{A}(x)$ and $\forall x\forall y(\mathfrak{A}(x)\wedge\mathfrak{A}(y)\vdash x=y)$ belong to $T$. $A(T)$ is defined to be the set of all the definite classes. Let $(\{x\}\mathfrak{A}(x))$ and $(\{x\}\mathfrak{B}(x))$ be two elements of $A(T)$. Then

$$(\{x\}\mathfrak{A}(x))\in^*_T(\{x\}\mathfrak{B}(x))$$

is defined to be

$$\text{"}\exists x\exists y(\mathfrak{A}(x)\wedge\mathfrak{B}(y)\wedge x\in y)\text{ belongs to }T\text{"}.$$

We see easily that Proposition 1 of [2] can be extended as follows:

Proposition. Let $T$ be a definite set theory and $b_1,\cdots,b_n$ be elements of $A(T)$ and represented by $\{x\}\mathfrak{B}_1(x),\cdots,\{x\}\mathfrak{B}_n(x)$ respectively. Then $\mathfrak{C}(b_1,\cdots,b_n)$ is satisfied in $<A(T),\ \in^*_T>$ if and only if

$$\exists x_1\cdots\exists x_n(\mathfrak{B}_1(x_1)\wedge\cdots\wedge\mathfrak{B}_n(x_n)\wedge\mathfrak{C}(x_1,\cdots,x_n))$$

belongs to $T$.

Let $a$ be a set in $C$, and let us consider the set theory with all the elements of $a$, consisting of all the formulas which are true in $C$. We denote it by $T_C(a)$ and write $B(a)$ in place of $B_{T_C(a)}$.

A set $b$ in $C$ is said to be *definable from* $a_0,\cdots,a_n$ *in* $C$, if there exists an element $\{x\}\mathfrak{A}(x,a_0,\cdots,a_n)$ of $B(\{a_0,\cdots,a_n\})$ such that $\mathfrak{A}(b,a_0,\cdots,a_n)$ and $\forall x\forall y(\mathfrak{A}(x,a_0,\cdots,a_n)\wedge\mathfrak{A}(y,a_0,\cdots,a_n)\vdash x=y)$ are satisfied.

A set $b$ is said to be *definable within* $a$, if $b$ is definable from elements of $a$. A set $a$ is called to be *definably closed*, if $a$ satisfies the following conditions:

1) $a$ is super-complete, i.e.

$$\forall x\forall y(x\in a\wedge(y\subseteq x\vee y\in x)\vdash y\in a)$$

is satisfied.

2) Any set definable within $a$ belongs to $a$.

Let us show that for an arbitrary set $a$ in $C$, there exists a set $a_0$ in $C$ which is definably closed and contains $a$ as an element. To define $a_0$, we shall further define some concepts. Since $<C,\ \in>$ is assumed to be regular, the rank $r(x)$ can be defined for all sets $x$ in the following way:

$r(0)$ is 0.

$r(x)$ is the least ordinal number $\alpha$ such that $\forall y(y\in x\vdash r(y)<\alpha)$. $D(a)$ is defined as follows: a set $b$ belongs to $D(a)$, if and only if $r(b)\leq r(c)$ holds for some $c$ which is definable from $a$. Let $D^{n+1}(a)$ mean $D(D^n(a))$ and let $a_0$ be $\bigcup_n D^n(a)$. This $a_0$ posesses the required property.

The use of rank may seem to be redundant in the construction above, but it simplifies the proof.

To prove (*) we shall first show that $T_C(a_0)$ is definite. For every element $\exists x\mathfrak{A}(x)$ of $T_C(a_0)$, consider the least ordinal number $\alpha$

such that

$$\mathfrak{E}x(r(x)=\alpha \wedge \mathfrak{A}(x)).$$

There exists a set $b$ such that $r(b)=\alpha \wedge \mathfrak{A}(b)$. Since $\alpha$ is definable within $a_0$, which is definably closed, $\alpha$ is an element of $a_0$. This implies that $b$ is also an element of $a_0$. Let $\{x\}\mathfrak{B}(x)$ be $\{x\}(x=b)$. Then $\mathfrak{E}x\mathfrak{B}(x)$, $\forall x \forall y(\mathfrak{B}(x) \wedge \mathfrak{B}(y) \vdash x=y)$ and $\mathfrak{E}x(\mathfrak{B}(x) \wedge \mathfrak{A}(x))$ are satisfied. Thus we see that $T_C(a_0)$ is definite.

From this consideration we see also that $<a_0, \in_{a_0}>$ is isomorphic to $<A(T_C(a_0)), \in^*_{T_C(a_0)}>$. Hence, by analogous arguments as in [2], we can conclude (*) in which $u$ and $x$ correspond to $a$ and $a_0$ respectively.

Let $R(\alpha)$ be the set consisting of all $x$ such that $r(x)<\alpha$. An ordinal number $\alpha$ is called to be an essentially inaccessible number, if $R(\alpha)$ is definably closed and $\alpha$ is inaccessible. Though this condition on $\alpha$ cannot be expressed in the first order predicate calculus with the only predicate $\in$, the following axiom of infinity seems to be a very interesting hypothesis:

$$\forall x \mathfrak{E}\alpha\ (x \in R(\alpha) \wedge \text{ '}\alpha\text{ is an essentially inaccessible number'}),$$

whence follows, for example,

$$\forall x \mathfrak{E}\alpha\ (x \in R(\alpha) \wedge \text{ '}\alpha\text{ is an inaccessible number'} \wedge \forall z(\mathfrak{A}(z) \vdash \mathfrak{D}(R(\alpha), z)),$$

where $\mathfrak{A}$ and $\mathfrak{D}$ have the same meanings as in [2].

## References

[1] K. Gödel: The Consistency of the Axiom of Choice and of the Generalized Continuum-hypothesis with the Axioms of Set Theory, Revised ed., Princeton (1951).

[2] G. Takeuti: Remarks on Cantor's absolute, I., J. Math. Soc. Japan, **13**, 197–206 (1961).