129. Re-topologization of Functional Space in Order that a Set of Operators will be Continuous

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The purpose of this note is to give a new topology to an abstract functional space in order that a set of linear operators, which need not be continuous primarily, will be continuous and that they will be extended to operators with the whole space as their domains. This may be regarded, in a sense, as an abstract generalization of the concept "distributions" by L. Schwartz [2].

This investigation has gotten the hint from the idea "negative norms" by P. D. Lax [1]. Here we shall give only general and abstract considerations. More concrete cases and applications will be given later on. A special case was considered by E. B. Cossi and the auther [3].

1. Let E be a *locally convex linear topological space.* We assume also that E is *bornologic.*

Let $\{T_\alpha\}$ $(\alpha\in\Omega)$ be a set of pre-closed linear operators from E into E, such that $D(T_\alpha)$, the domain of T_α, is dense in E. We assume also that the identical operator $1\in\{T_\alpha\}$.

Theorem 1. *Let T'_α be the adjoint operator of T_α and put $F=\bigcap_\alpha D(T'_\alpha)$. Assume that F is total on E, i.e., if $x\in E$ and $f(x)=0$ for all $f\in F$ then $x=0$. Then we can give a new topology to E, in such a way that all T_α will be continuous from E with the primary topology into E with the new topology.*

Proof: First let us give a new topology to F, in such a way that all T'_α will be continuous on $\tilde{F}$ into E', the dual of E, where $\tilde{F}$ means the same set as F but with the new topology.

For this purpose define the semi-norm $p'_{A,\alpha}$ on E' by

$$p'_{A,\alpha}(f)=\sup\{|T'_\alpha f(x)|;x\in A\} \qquad \text{for } f\in F,$$

where A is any bounded set in E. Then by the set of semi-norms $\{p'_{A,\alpha}\}$ $(A\in B(E),\ \alpha\in\Omega)$[1] F becomes a locally convex linear topological space $\tilde{F}$. As $1\in\{T'_\alpha\}$, the new topology of $\tilde{F}$ is stronger than the old topology of $F\subset E'$.

The operators T'_α are continuous on $\tilde{F}$ into E'. Because the topology of E' is given by the system of semi-norms $\{p'_A\}$ $(A\in B(E))$ defined by

$$p'_A(f)=\sup\{|f(x)|;x\in A\}$$

1) $B(E)$ denotes the system of all bounded sets in E.

and

$$p'_A(T'_\alpha f)=p'_{A,\alpha}(f).$$

Now we define the new semi-norm $\tilde{p}_B(x)$ on E by

$$\tilde{p}_B(x)=\sup\{|f(x)|; f\in B\} \qquad \text{for } x\in E,$$

where B is any bounded set in $\tilde{F}$, $B\in B(\tilde{F})$. By the system of all semi-norms $\{\tilde{p}_B\}$ ($B\in B(\tilde{F})$), E becomes a locally convex linear topological space $\hat{E}$.[2]

All T_α are continuous from E (on $D(T_\alpha)$) into $\hat{E}$. Because, $T'_\alpha B$ is bounded in E' for any $B\in B(\tilde{F})$ and

$$\tilde{p}_B(T_\alpha x)=\sup\{|f(x)|; f\in T'_\alpha B\},$$

so that $\tilde{p}_B(T_\alpha x)$, as a function of x, is bounded on any bounded subset of $D(T_\alpha)$ in E. Therefore T_α is continuous from E into $\hat{E}$, since E is bornologic. Q.E.D.

Now let $\tilde{E}$ be the completion of the linear topological space $\hat{E}$. Then the operator T_α can be extended uniquely to a continuous linear operator on E into $\tilde{E}$, since $D(T_\alpha)$ is dense in E. The topology of $\tilde{E}$ is weaker than that of E, since the canonical mapping of E into $\tilde{E}$ is continuous.

If E is a Banach space and $\{T_\alpha\}$ is a finite set of operators, then $\tilde{E}$ is also a Banach space.

2. Theorem 2. *Besides the previous assumptions we assume also that $\{T_\alpha\}$ ($\alpha\in\Omega$) forms an operator algebra[3] on D into D, where D is a linear set dense in E. Let $\hat{D}$ be the same set as D provided with the topology induced by that of $\tilde{E}$. Then T_α are continuous on $\hat{D}$ into $\hat{D}$, and can be extended uniquely to continuous linear operators on $\tilde{E}$ into $\tilde{E}$.*

Proof: T'_α are continuous on $\tilde{F}$ into $\tilde{F}$. Because, $T'_{\alpha'}T'_\alpha=T'_{\alpha''}$ ($\alpha''\in\Omega$) for any α, $\alpha'\in\Omega$ and

$$p'_{A,\alpha'}(T'_\alpha f)=p'_{A,\alpha''}(f).$$

Thus, $T'_\alpha B$ is bounded in $\tilde{F}$ for any $B\in B(\tilde{F})$ and

$$\tilde{p}_B(T_\alpha x)=\tilde{p}_{T'_\alpha B}(x).$$

Hence T_α is continuous on $\hat{D}$ into $\tilde{E}$.

As D is dense in E and E is dense in $\tilde{E}$, D is dense in $\tilde{E}$. Therefore T_α can be extended uniquely to a continuous linear operator on $\tilde{E}$ into $\tilde{E}$.

3. With the same notations and assumptions as in 1, now we

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2) $\tilde{p}_B(x)=0$ for all $B\in B(\tilde{F})$ implies $x=0$, since F is total on E.

3) For any $\alpha, \alpha'\in\Omega$ there exists $\alpha''\in\Omega$ such that $T_\alpha T_{\alpha'}=T_{\alpha'}'$.

shall give simple theorems on the duality of $\tilde{E}$ and $\tilde{F}$.

Theorem 3. *If $\tilde{F}$ is reflexive,*[4] *then $\tilde{E}=(\tilde{F})'$ and $(\tilde{E})'=\tilde{F}$.*

Proof: We can easily see that $\tilde{E}\subset(\tilde{F})'$ by the definition of $\tilde{E}$, since $\tilde{F}$ is bornologic and $(\tilde{F})'$ is complete. If it was the case $\tilde{E}\neq(\tilde{F})'$, then, as $\tilde{E}$ is closed in $(\tilde{F})'$, there would exist $\phi\in(\tilde{F})''$ such that

i) $\phi(x)=0$ for all $x\in\tilde{E}(\subset E)$,

and ii) $\phi(\psi)\neq 0$ for some $\psi\in(\tilde{F})'$.

As $\tilde{F}$ is reflexive, we can assume $\phi\in\tilde{F}\subset E'$, hence $\phi=0$ by i), and $\phi(\psi)=\psi(\phi)=0$ which contradicts with ii).

Further we get easily $(\tilde{E})'=\tilde{F}$.

Theorem 4. *If E is reflexive,*[5] *then $\tilde{E}=(\tilde{F})'$ and $(\tilde{E})'=\tilde{F}$.*

Proof: By Theorem 3 we have to show that $\tilde{F}$ is reflexive. Since E', the dual of E, is reflexive, the product linear topological space $E'\times\prod_{\alpha\in\Omega}E'_\alpha$ with the weakest topology, where $E_\alpha=E$, is reflexive. The graph $\{(f, T'_\alpha f)(\alpha\in\Omega); f\in F\}$ is a closed linear subspace of $E'\times\prod_\alpha E'_\alpha$ and is homeomorphically isomoph to $\tilde{F}$. Hence $\tilde{F}$ is reflexive.

References

[1] Lax, P. D.,: On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations, Comm. Pure and Appl. Math. vol. 8, 615-633 (1955).

[2] Schwartz L.,: Theorie des Distributions, Paris, Hermann, 1950-1951.

[3] Nagumo M., and Cossi E. B.,: A note on closed linear operator, to appear in Summa Brasiliensis Mathematicae.

[4] We assume also that $\tilde{F}$ is bornologic. (Added in proof.)

[5] We assume also that E' is bornologic. (Added in proof.)