127. Remarks on Metric Spaces with U-extension Properties

By Masahiko ATSUJI
Senshu University, Tokyo

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In this paper, a space S is uniform and a function is real and, unless otherwise specified, uniformly continuous. A sequence $\{A_n\}$ of subsets is said to be *discretely normally separated* by a sequence $\{B_n\}$ of subsets if $\{B_n\}$ is discrete and, for every n, there is a function f, $0\leq f\leq 1$, on S with the value 1 on A_n and 0 on $S-B_n$; $\{A_n\}$ is *uniformly separated* by $\{B_n\}$ if there is an entourage U with $B_n\supset U(A_n)$ for every n; $\{A_n\}$ is *U-discrete*, U being an entourage, if, for any point $x\in S$, $U(x)$ meets at most one member of the sequence; finally, $\{A_n\}$ is *uniformly discrete* if it is U-discrete for some U.

0. Let us give our attention to the similarity between the following two conditions:

(UC) *Any sequence $\{A_n\}$ of subsets discretely normally separated by some sequence $\{B_n\}$ of subsets is uniformly separated by $\{B_n\}$*, which is a necessary and sufficient condition in order for any continuous real function on a space to be uniform [1, Theorem 1].

(E) *Let $\{A_n\}$ be a U-discrete sequence of subsets, and $\{a_n\}$ a sequence of natural numbers, then there is an entourage V with $V^{a_n}(A_n)\subset U(A_n)$ for every n*, which is a necessary and sufficient condition in order for any function defined on any uniform subspace of a space S to have a uniform extension to S [2]. When $\{A_n\}$ is a U-discrete sequence and $W^4\subset U$, then $\{A_n\}$ is discretely normally separated by the sequence $\{W(A_n)\}$.

In a metric space, we have several conditions equivalent to (UC) [3, Theorem 1], while we have a condition corresponding to (E) in which V^{a_n} in (E) is replaced by $V^\infty=\bigcup_m V^m$ [4, Lemma 2]. Therefore it is natural to seek the conditions of metric spaces equivalent to (E) corresponding to each of the known conditions equivalent to (UC).

1. From here on, we will consider a space metric and complete (a metric space satisfying (UC) is necessarily complete). $V_{1/n}$ is the entourage of the space consisting of pairs of points whose distances apart are less than $1/n$, a V_e-discrete family, e being a positive number, is simply said to be *e-discrete*, and a space satisfying (E) is said to have a *u-extension property*. We know [4, Theorem 2] that *a space S has a u-extension property if and only if for any natural number n there is a compact subset K such that, for any open subset G containing K, there is a natural number m satisfying*

$V_{1/n}(x)\supset V^{\infty}_{1/m}(x)$ *for every* $x \notin G$. The compact set K in this statement is given by

$$K=\frown \overline{K}_i,\ K_i=\{x;\ V_{1/n}(x)\not\supset V^{\infty}_{1/i}(x)\}.$$

Now we are going to show that *if the space S has a u-extension property, then K in the above statement can be taken as* $\frown K_i$. Take a point p in $\frown \overline{K}_i$, then, for any natural number i and a positive e, there is a point q_i in K_i satisfying

(*) $$d(p,q_i)<e,\ V_{1/n}(q_i)\not\supset V^{\infty}_{1/i}(q_i).$$

Suppose that $V_{1/n}(p)\supset V^{\infty}_{1/j}(p)$ for some j, then, for all $i\geq j$,

(**) $$V_{1/n}(p)\supset V^{\infty}_{1/i}(p).$$

If $1/n>e'=\sup\{d(p,x);\ x\in V^{\infty}_{1/i}(p)\}$ for some $i\geq j$, then, for $e>0$ with $1/n-e'>e<1/i$, there is a point q_i satisfying (*); while, for every $x\in V^{\infty}_{1/i}(q_i)=V^{\infty}_{1/i}(p)$, we have $d(q_i,x)\leq d(q_i,p)+d(p,x)<e+e'<1/n$, which contradicts (*). Therefore $1/n=\sup\{d(p,x);\ x\in V^{\infty}_{1/i}(p)\}$ for every $i\geq j$. Since $d(p,x)<1/n$ for every $x\in V^{\infty}_{1/i}(p)$, there are an infinite number of points $x^i_1, x^i_2,\cdots$ in $V^{\infty}_{1/i}(p)$ such that $d(p,x^i_k)$ converges to $1/n$. As the space, and so $V^{\infty}_{1/i}(p)$, is complete, we may assume the uniform discreteness of $\{x^i_k\}$. There is a compact set H such that, for any open set G containing H, there is a natural number m satisfying $V_{1/(n+1)}(q)\supset V^{\infty}_{1/m}(q)$ for all $q\notin G$, then, since $p\in V^{\infty}_{1/i}(x^i_k)$ for $i\geq j$, the intersection A of H with the closure B of $\{x^i_k;\ i\geq j,\ k=1,2,\cdots\}$ is not empty. If A includes points $y_1, y_2,\cdots$ such that $d(p,y_k)$ converge to $1/n$, then it also includes a point y with $d(p,y)=1/n$. Let i' be any natural number $\geq j$, and e'' any positive number, then there is a positive number $b<e''$ for which $V_b(y)$ does not include any x^h_k, $h<i', k=1,2,\cdots$, because $y\neq x^i_k$ for each $i\geq j,\ k=1,2,\cdots$ and $\{x^h_k;\ k=1,2,\cdots\}$ is uniformly discrete for every h. Therefore, since $y\in A \subset B$, $V_b(y)$ includes some $x^i_k, i\geq i'$, which belongs to $V^{\infty}_{1/i}(p)\subset V^{\infty}_{1/i'}(p)$, and thus we have $y\in V^{\infty}_{1/i'}(p)$, which contradicts (**). Consequently, there is a positive number $c<1/n$ such that $d(p,y)<c$ for all $y\in A$. Let us take a neighborhood $U(y)$ of $y\in A$ satisfying $d(p,z)<c$ for all $z\in U(y)$, and a neighborhood $U(y)$ of $y\in H-A$ satisfying $U(y)\frown B=\phi$, then we find a natural number $m\geq j$ satisfying $V_{1/(n+1)}(q)\supset V^{\infty}_{1/m}(q)$ for every $q\notin G=\bigcap_{y\in H} U(y)$. However, there is k with $d(p,x^m_k)>\max(c, 1/(n+1))$; since $d(p,x^m_k)>c$, x^m_k cannot belong to any $U(y)$ of $y\in A$; and since $x^m_k\in B$, x^m_k cannot belong to any $U(y)$ of $y\in H-A$, i.e. $x^m_k\notin G$, a contradiction. Consequently, we have $V_{1/n}(p)\not\supset V^{\infty}_{1/i}(p)$ for every i, i.e. $p\in\frown K_i$. If we say a point p is *n-point* when $V_{1/n}(p)\not\supset V^{\infty}_{1/i}(p)$ for every i, then here we have a better result than Theorem 2 of [4]: *a complete metric space has a u-extension property if and only if, for any natural number n, the set K of all n-points is compact, and for any open set G containing K there is a natural number m satisfying* $V_{1/n}(x)\supset V^{\infty}_{1/m}(x)$ *for all* $x\notin G$.

2. Lemma 2 of [4] shows that *if a space S has a u-extension property, then, for any natural number n and any uniformly discrete infinite set D, there is a natural number m such that $V_{1/n}(x) \supset V_{1/m}^{\infty}(x)$ in S for all but a finite number of points x in D.* We can readily see that this condition is also sufficient for a u-extension property. In fact, using only this condition (without the direct use of the hypothesis of u-extension property), we have verified the "only if" part of Theorem 2 of [4].

3. Moreover, in the proof of the "only if" part of Theorem 2 of [4], we used Lemma 2 only for a countable set D, so *a space S has a u-extension property if and only if, for any natural number n and any uniformly discrete sequence of points, there is a natural number m such that $V_{1/n}(x) \supset V_{1/m}^{\infty}(x)$ in S for all but a finite number of points x in the sequence.*

4. *A space S has a u-extension property if and only if, for any function f defined on any uniform subspace A of S, and for any natural number n, there are natural numbers m and m' such that $V_{1/n}(x) \supset A \frown V_{1/m}^{\infty}(x)$ for every point $x \in B = \{x \in A; |f(x)| \geq m'\}$.* In fact, let f be defined on A in S with a u-extension property. There is a natural number n' such that $d(x, y) < 1/n'$ implies $|f(x) - f(y)| < 1$ on A. For each n, there is a compact K and an m with $V_{1/n}(x) \supset V_{1/m}^{\infty}(x)$ in S for each $x \notin G = V_{1/n'}(K)$. We have $|f(x)| < m'$ on $G \frown A$ for some m', and so we have $x \notin G$ whenever $x \in \{x \in A; |f(x)| \geq m'\}$. Conversely, suppose that f is defined on a subspace A of S satisfying the condition in the above statement, then there is n such that $d(x, y) < 2/n$ implies $|f(x) - f(y)| < 1$ on A, and, for this n, there are $m \geq n$ and m' satisfying this condition. $\{S - \smile_{x \in B} V_{1/m}^{\infty}(x),\ V_{1/m}^{\infty}(x);\ x \in B\}$ is a uniform decomposition of S, and $y, z \in A \frown V_{1/m}^{\infty}(x)$ implies $d(y, z) < 2/n$ and $|f(y) - f(z)| < 1$. By Corollary 1 in [4], f can be uniformly extended to S. We can replace "$V_{1/n}(x) \supset A \frown V_{1/m}^{\infty}(x)$" with "$V_{1/n}(x) \supset V_{1/m}^{\infty}(x)$ in S" in this statement, and then we have a simpler proof by using the statement in §3. However, the latter condition demands too much when we want to apply the statement to get the uniform extension of some function on some subspace of S. For instancc, consider the subspace $A = \smile_{n>3}\{[n, n+1/n^2] \smile [n+1/2, n+3/4]\}$ of the space $S = \smile_{n>3}\{[n, n+1/4] \smile [n+1/2, n+3/4]\}$ and the function $f(x) = x^2$ on $[n, n+1/n^2]$ and $= 0$ on $[n+1/2, n+3/4]$; neither S nor A has a u-extension property, f can be uniformly extended to S, $V_{1/5}(x) \not\supset V_{1/m}^{\infty}(x)$ in S for x in any B and any m, and $V_{1/n}(x) \supset V_{1/m}^{\infty}(x) \frown A$ for any n and x in some B and some m.

5. From §4 we have immediately the statement (cf. [4, Corollary 1]) that *a space S has a u-extension property if and only if, for any uniform subspace A of S and for any function f on A, there*

is a uniform decomposition $\{S_\alpha\}$ *of* S *such that the diameters of* $f(S_\alpha \frown A)$*'s are less than a positive number.*

6. *A space* S *has a u-extension property if and only if, for any sequences* $\{A_i\}$ *and* $\{B_i\}$ *of subsets with the properties* $V_{1/n}(\smile A_i) \frown(\smile B_i)=\phi$ *and* $\underset{h}{\frown} V_{1/n}(A_{i_h} \smile B_{i_h})=\phi$ *for a fixed natural number* n *and any subsequences* $\{A_{i_h}\}$ *and* $\{B_{i_h}\}$*, we have* $V_{1/m}^{\infty}(A_i) \frown V_{1/m}^{\infty}(B_i)=\phi$ *for some* m *and* i. In fact, suppose that $V_{1/m}^{\infty}(A_i) \frown V_{1/m}^{\infty}(B_i)$ includes a point for any m and any i, then there are points $a_i \in A_i$ and $b_i \in B_i$ which can be bound by a $1/m$-chain to each other. For any finite set F, we can take a_i and b_i satisfying $F \frown V_{1/n}(a_i \smile b_i)=\phi$ because of $\underset{h}{\frown} V_{1/n}(A_{i_h} \smile B_{i_h})=\phi$ for any infinite subsequence $\{i_h\}$ of $\{i\}$. Thus we have $1/2n$-discrete sequences $\{a_i\}$ and $\{b_i\}$ of points such that a_i and b_i can be bound by a $1/i$-chain, and $V_{1/n}(\{a_i\}) \frown \{b_i\}=\phi$. By §3, the existence of these sequences is impossible in a space with a u-extension property. Conversely, if there is a uniformly discrete sequence $\{x_i\}$ such that, for some n' and any m', there is a point included in $V_{1/m'}^{\infty}(x_i)$ but not in $V_{1/n'}(x_i)$ for an infinite number of i's, then there is an $8/n$-discrete sequence $\{x_{i_h}\}$ for some n and a sequence $\{y_{i_h}\}$ such that $1/n<d(x_{i_h}, y_{i_h})<2/n$ and x_{i_h} and y_{i_h} can be bound by a $1/(n+h)$-chain. For $A_j=\{x_{i_h};\, h \geq j\}$ and $B_j=\{y_{i_h};\, h \geq j\}$, we cannot find m and j with $V_{1/m}^{\infty}(A_j) \frown V_{1/m}^{\infty}(B_j)=\phi$.

Summarizing these results, we have (cf. [3, Theorem 1])

Theorem. *The following properties of a complete metric space* S *are equivalent.*

(0) *S has a u-extension property.*

(1) *For any natural number* n *and any uniformly discrete sequence of points, there is a natural number* m *such that* $V_{1/n}(x) \supset V_{1/n}{}^{\infty}(x)$ *in* S *for all but a finite number of points* x *in the sequence.*

(2) *For any natural number* n *and any uniformly discrete infinite set* D*, there is a natural number* m *such that* $V_{1/n}(x) \supset V_{1/m}{}^{\infty}(x)$ *in* S *for all but a finite number of points* x *in* D.

(3) *For any natural number* n*, the set* K *of all* n*-points is compact, and, for any open set* G *containing* K*, there is a natural number* m *satisfying* $V_{1/n}(x) \supset V_{1/m}^{\infty}(x)$ *for every* $x \notin G$.

(4) *Let* $\{A_i\}$ *and* $\{B_i\}$ *be any sequences of subsets with the property that there is a natural number* n *such that* $V_{1/n}(\smile A_i) \frown(\smile B_i)=\phi$ *and* $\underset{h}{\frown} V_{1/n}(A_{i_h} \smile B_{i_h})=\phi$ *for any subsequence* $\{i_h\}$ *of the sequence* $\{i\}$ *of indices, then* $V_{1/m}{}^{\infty}(A_i) \frown V_{1/m}{}^{\infty}(B_i)=\phi$ *for some* m *and* i.

(5) *For any function* f *defined on any uniform subspace* A *of* S *and for any natural number* n*, there are natural numbers* m *and* m' *such that* $V_{1/n}(x) \supset V_{1/m}{}^{\infty}(x) \frown A$ *for every point* $x \in \{x \in A;\, |f(x)| \geq m'\}$.

(6) *For any uniform subspace* A *of* S *and any function* f *on*

A, there is a uniform decomposition $\{S_\alpha\}$ of S such that the diameters of $f(S_\alpha \frown A)$'s are less than a positive number.

References

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