140. Some Characterizations of Fourier Transforms. II

By Koziro IWASAKI
Musashi Institute of Technology, Tokyo

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1. In the theory of the Fourier exponential transform on the real number field $\boldsymbol{R}$ the following four properties play important roles. Namely,

a) the Fourier exponential transform

$$E: \varphi(x) \rightarrow E\varphi(x) = \int_{-\infty}^{\infty} e^{2\pi i x t} \varphi(t)\, dt$$

is a linear mapping from $\mathfrak{P}$ onto itself where $\mathfrak{P}$ is the space of all functions of class C^∞ whose derivatives are all rapidly decreasing,

b) $E(\varphi * \psi) = E\varphi \cdot E\psi$,

c) $\int_{\boldsymbol{R}} |E\varphi|^2 dx = \int_{\boldsymbol{R}} |\varphi|^2 dx$,

d) $\sum_{n\in \boldsymbol{Z}} E\varphi(n) = \sum_{n \in \boldsymbol{Z}} \varphi(n)$

where φ and ψ belong to $\mathfrak{P}$, $\varphi * \psi$ is the convultion of φ and ψ, and $\boldsymbol{Z}$ is the set of all integers.

Some years ago we have pointed out that the properties b) and d) characterize the Fourier exponential transform ([2]). In this paper we shall deal with another characterization. We denote $\varphi(x+a)$ with $\varphi_a(x)$ as a function of x.

Now the main result is as follows:

Theorem. *If there exists a linear mapping T from $\mathfrak{P}$ into the space of C^∞ functions on a Riemannian manifold $\mathfrak{R}$ satisfying the conditions:*

I) *when a function series $\varphi_1, \varphi_2, \cdots$ in $\mathfrak{P}$ converges to 0 by L^1-topology, the series $T\varphi_1, T\varphi_2, \cdots$ converges to 0 by L^∞-topology,*

II_1) *to any point ξ of $\mathfrak{R}$ and any open set U containing ξ there exists a function φ in $\mathfrak{P}$ such that the support of $T\varphi$ is contained in U and $T\varphi(\xi)$ is different from 0 and*

II_2) *to the same function φ $T\varphi_a(\xi)$* grad $T\varphi(\xi)$ *differs from $T\varphi(\xi)$* grad $T\varphi_a(\xi)$ *with some real number a* (here a may depend on φ),

III) $$T(\varphi * \psi) = T\varphi \cdot T\psi,$$

IV) $$\int_{\boldsymbol{R}} |T\varphi|^2 d\xi = \int_{\boldsymbol{R}} |\varphi|^2 dx,$$

then there is a C^∞ bijection r from $\mathfrak{R}$ to $\boldsymbol{R}$ such that

$$T\varphi(\xi) = E\varphi(r\xi).$$

Moreover if we assume an additional hypothesis

V) *there is a discrete subset* $\mathfrak{Z}$ *of* $\mathfrak{R}$ *to which* $\sum_{\nu\in\mathfrak{Z}} T\varphi(\nu)$ *is absolutely convergent and is equal to* $\sum_{n\in \boldsymbol{Z}}\varphi(n)$ *for any* φ *in* $\mathfrak{P}$, *then*

$$r(\mathfrak{Z})=\boldsymbol{Z}.$$

2. At first we shall prove several lemmas under the hypotheses I, II_1, II_2, III and IV. We denote $\dfrac{T\varphi_a(\xi)}{T\varphi(\xi)}$ with $\xi(a)$ for φ in $\mathfrak{P}$, a in $\boldsymbol{R}$ and ξ in $\mathfrak{R}$ if $T\varphi((\xi)\neq 0$.

Lemma 1. *$\xi(a)$ is independent of the choice of φ and*

$$\xi(a+b)=\xi(a)\xi(b).$$

Proof. Let $\psi\in\mathfrak{P}$ and $T\psi(\xi)\neq 0$. We have $(\varphi*\psi)_a=\varphi_a*\psi=\varphi*\psi_a$ and $T(\varphi*\psi)(\xi)=T\varphi(\xi)T\psi(\xi)\neq 0$ by the hypothesis III. Then

$$\frac{T(\varphi*\psi)_a}{T(\varphi*\psi)}=\frac{T\varphi_a\cdot T\psi}{T\varphi\cdot T\psi}=\frac{T\varphi\cdot T\psi_a}{T\varphi\cdot T\psi}.$$

Because $\|\varphi_a-\varphi(x)\|_{L^1}<\varepsilon$ if a is small enough $T\varphi_a(\xi)\neq 0$ for small a by the Hypothesis I.

And the equation $\dfrac{T\varphi_{a+b}(\xi)}{T\varphi(\xi)}=\dfrac{T(\varphi_a)_b(\xi)}{T\varphi_a(\xi)}\cdot\dfrac{T\varphi_a(\xi)}{T\varphi(\xi)}$ has a meaning. Or

$\xi(a+b)=\xi(a)\xi(b)$ for sufficiently small a.

Now we can easily prove this equation for arbitrary a and b. Q.E.D.

Corollary 1. *For every a $\xi(a)\neq 0$ and $T\varphi_a(\xi)\neq 0$ if $T\varphi(\xi)\neq 0$.*

Corollary 2. *$\xi(a)$ is continuous with respect to ξ.*

Lemma 2. $$|\xi(a)|=1$$ *for every a in $\boldsymbol{R}$ and ξ in $\mathfrak{R}$.*

Proof. By Hypothesis IV and lemma 1

$$\int_{\mathfrak{R}}|T\varphi_a|^2\,d\xi=\int_{\boldsymbol{R}}|\varphi_a|^2\,dx=\int_{\boldsymbol{Z}}|\varphi|^2\,dx=\int_{\mathfrak{R}}|T\varphi|^2\,d\xi=\int_{\mathfrak{R}}|\xi(a)|^2|T\varphi|^2\,d\xi.$$

If $|\xi_0(a)|>1$ for ξ_0 in $\mathfrak{R}$ then $|\xi(a)|>1$ for any point ξ in some neighbourhood U of ξ_0 by the corollary 2 of Lemma 1. According to Hypothesis II_1 there is a function $T\varphi$ in $T\mathfrak{P}$ different from 0 whose carrier is contained in U. For such $T\varphi$

$$\int_{\mathfrak{R}}|\xi(a)|^2|T\varphi(\xi)|^2\,d\xi>\int_{\mathfrak{R}}|T\varphi(\xi)|^2\,d\xi.$$

Thus we have arrived at a condition.

Corollary. *For any ξ in $\mathfrak{R}$ there exists a real number $r(\xi)$ such that* $$\xi(a)=\exp(-2\pi i r(\xi)a).$$

Lemma 3. *If we assume Hypothesis* V, *besides* I, II_1, II_2, III *and* IV, *then $T\varphi(\nu)=E\varphi(r(\nu))$ for any ν in $\mathfrak{Z}$ and $r(\nu)$ is an integer.*

Moreover $\nu\to r(\nu)$ is a bijection from $\mathfrak{Z}$ to $\boldsymbol{Z}$.

Proof. By the hypotheses III, V and Lemma 1

$$\sum_{\nu\in\mathfrak{Z}}T(\varphi*\psi)(\nu)=\sum_{\nu\in\mathfrak{Z}}T\varphi(\nu)\cdot T\psi(\nu)$$

$$=\sum_{n\in\boldsymbol{Z}}\varphi*\psi(n)=\sum_{n\in\boldsymbol{Z}}\int_{\boldsymbol{R}}\varphi(n-x)\psi(x)\,dx$$

$$=\int_{R}\sum_{n\in Z}\varphi_{-x}(n)\,\psi(x)\,dx=\int_{R}\sum_{\nu\in\mathfrak{Z}}T\varphi_{-x}(\nu)\,\psi(x)\,dx$$
$$=\int_{R}\sum_{\nu\in\mathfrak{Z}}T\varphi(\nu)\exp(2\pi i r(\nu)x)\,\psi(x)\,dx$$
$$=\sum_{\nu\in\mathfrak{Z}}T\varphi(\nu)\,E\psi(r(\nu))$$

for any φ in $\mathfrak{P}$ and ψ in $\mathfrak{P}$ with compact support. If we choose as $T\varphi$ such a function that $T\varphi(\nu)\neq 0$ and $T\varphi(\nu')=0$ for the other elements in $\mathfrak{Z}$ than ν we get $T\varphi(\nu)T\psi(\nu)=T\varphi(\nu)E\psi(r(\nu))$. Therefore $T\psi(\nu)=E\psi(r(\nu))$ for any ν in $\mathfrak{Z}$ and ψ in $\mathfrak{P}$ with compact support i.e. the element of $\mathfrak{D}$. Since $\mathfrak{D}$ is dense in $\mathfrak{P}$ with L^1-topology we have proved

(1) $\quad T\psi(\nu)=E\psi(r(\nu))$ for any ψ is $\mathfrak{P}$.

By Hypothesis V and Lemma 2 we get

$$\sum_{\nu\in\mathfrak{Z}}T\varphi_m(\nu)=\sum_{n\in Z}\varphi_m(n)=\sum_{n\in Z}\varphi(n)=\sum_{\nu\in\mathfrak{Z}}T\varphi(\nu)$$
$$=\sum_{\nu\ni\mathfrak{Z}}\exp(-2\pi i r(\nu)m)\,T\varphi(\nu)\ \text{ for } m\in Z.$$

By the same choice of $T\varphi$ as in the proof of (1) we have $\exp(-2\pi i r(\nu)m)=1$ for all integers m. This means that $r(\nu)$ is an integer.

Substituting (1) in V

$$\sum_{\in\mathfrak{Z}}E\varphi(r(\nu))=\sum_{n\in Z}\varphi(n)=\sum_{n\in Z}E\varphi(n).$$

Again choosing $E\varphi$ in the similar manner we can prove that $\nu\to r(\nu)$ is a bijection.

Lemma 4. *$r(\xi)$ is of class C^∞ as a function of ξ.*

Proof. By the definition and previous lemmas

$$\exp(-2\pi i r(\xi))=\frac{T\varphi_1(\xi)}{T\varphi(\xi)}$$

and $T\varphi(\xi)$, $T\varphi_1(\xi)$ are of class C^∞ with respect to ξ. Q.E.D.

Lemma 5. *$\int_{\mathfrak{R}}T\varphi\cdot\overline{T\psi}\,d\xi=\int_{\mathfrak{R}}\varphi\cdot\bar{\psi}\,dx$ for any φ and ψ in $\mathfrak{P}$.*

(Evident.)

Lemma 6. $$\overline{T\varphi}=T\hat{\varphi}$$

where $$\hat{\varphi}(x)=\overline{\varphi(-x)}.$$

Proof. By the Hypotheses IV and III

$$\int_{R}|\varphi*\psi|^2\,dx=\int_{\mathfrak{R}}|T(\varphi*\psi)|^2\,d\xi=\int_{\mathfrak{R}}|T\varphi|^2\,|T\psi|^2\,d\xi$$
$$=\int_{R\times R\times R}\varphi(x-u)\psi(u)\overline{\varphi(x-t)}\,\overline{\psi(t)}\,du\,dt\,dx$$
$$=\int_{R\times R}\varphi*\hat{\varphi}(t-u)\,\psi(u)\,\overline{\psi(t)}\,du\,dt=\int_{R}\varphi*\hat{\varphi}*\psi(t)\,\overline{\psi(t)}\,dt$$
$$=\int_{\mathfrak{R}}T(\varphi*\hat{\varphi}*\psi)\overline{T\psi}\,d\xi\quad(\text{Lemma 5})=\int_{\mathfrak{R}}T\varphi T\hat{\varphi}\,|T\psi|^2\,d\xi.$$

Thus we get $$\int_{\mathfrak{R}}|T\varphi|^2\,|T\psi|^2\,d\xi=\int_{\mathfrak{R}}T\varphi\,T\hat{\varphi}\,|T\psi|^2\,d\xi.$$

If $T\varphi T\hat{\varphi}$ is not real at ξ_0 there exists a neighbourhood U of ξ_0 where $\mathfrak{I}(T\varphi(\xi)T\hat{\varphi}(\xi))$ has the same sign as $\mathfrak{I}(T\varphi(\xi_0)T\hat{\varphi}(\xi_0))$ and if $T\varphi T\hat{\varphi}$ is real on $\mathfrak{R}$ and differs from $|T\varphi|^2$ at ξ_0 there is a neighbourhood U of U of ξ_0 where $T\varphi(\xi)T\hat{\varphi}(\xi)-|T\varphi(\xi)|^2$ has the same sign as $T\varphi(\xi_0)T\hat{\varphi}(\xi_0) -|T\varphi(\xi_0)|^2$. Using $T\psi$ with $T\psi(\xi_0)\neq 0$ whose support is contained in U, we arrive at a contradiction. Therefore $|T\varphi|^2=T\varphi\cdot T\hat{\varphi}$. Now if $T\varphi(\xi)\neq 0$ then $\overline{T\varphi(\xi)}=T\hat{\varphi}(\xi)$ and if $T\hat{\varphi}(\xi)\neq 0$ then $T\hat{\varphi}(\xi)=T\hat{\hat{\varphi}}(\xi)=T\varphi(\xi)$. Finally if $T\varphi(\xi)=0$ clearly $T\hat{\varphi}(\xi)=0$.

Lemma 7. *There are non-negative C^∞ functions on $\boldsymbol{R}$ $\alpha_1, \alpha_2, \cdots$ with compact support such that to any function φ in $\mathfrak{P}$, the series $\varphi*\alpha_1, \varphi*\alpha_2, \cdots$ converges to φ pointwise and in L^1-topology.*

(Schwartz [3] tome II, pp. 22 and 23.)

Lemma 8. $\int_{\mathfrak{R}} T\varphi d\xi=\varphi(0)$ *for φ in $\mathfrak{P}$ such that $T\varphi$ has the compact support.*

Proof. By Lemmas 5, 6, and Hypothesis III

$$\int_{\mathfrak{R}} T\varphi\cdot \overline{T\psi}\, d\xi=\int_{R} \varphi\cdot\bar{\psi}\, dx=\varphi*\hat{\psi}(0)=\int_{\mathfrak{R}} T\varphi\cdot T\hat{\psi}\, d\xi=\int_{\mathfrak{R}} T(\varphi*\hat{\psi})\, d\xi.$$

If we substitute in $\hat{\psi}$ $\alpha_1, \alpha_2, \cdots$ in Lemma 7, we get as the limit

$$\int_{\mathfrak{R}} T\varphi\, d\xi=\psi(0).$$

Lemma 9. *For any point ξ in $\mathfrak{R}$ we can take a local coordinates system having $r(\xi)$ as one of its coordinates.*

Proof. By Corollary of Lemma 2, Corollaries of Lemma 1 and Hypothesis II_2

$$2\pi ia \operatorname{grad} r(\xi)=\operatorname{grad}\left(\log\frac{T\varphi_a(\xi)}{T\varphi(\xi)}\right)$$
$$=\frac{1}{T\varphi_a(\xi)}\operatorname{grad} T\varphi_a(\xi)-\frac{1}{T\varphi(\xi)}\operatorname{grad} T\varphi(\xi)\neq 0. \quad \text{Q.E.D.}$$

3. Let U be a relatively compact open set in $\mathfrak{R}$ in which a local coordinate system $\xi^1,\cdots\xi^n$, where $\xi^1=r(\xi)$, is admissible. Take a function φ in $\mathfrak{P}$ different from 0 such that the support of $T\varphi$ is contained in U. Now, we apply Lemma 8 to $\varphi_a(x)$:

$$\varphi(a)=\varphi_a(0)=\int_{\mathfrak{R}} T\varphi_a\, d\xi=\int_U \exp(-2\pi ia\,\xi^1)\, T\varphi(\xi)\, d\xi.$$

By Lemma 9 we get, with positive function $g(\xi)$,

$$\varphi(a)=\int_{-\infty}^{\infty}\exp(-2\pi ia\xi^1)\Big(\int_{U|\xi^1} T\varphi\cdot g(\xi) d\,\xi^2\cdots d\xi^n\Big)\, d\xi^1$$

for any real number a. And by the inversion theorem of Fourier transform (Bochner and Chandrasekharan [1] p. 10)

$$\int_{r(\xi)=x} T\varphi\cdot g(\xi)\, d\xi^2\cdots d\xi^n=E\varphi(x).$$

Because $T(\underbrace{\varphi*\cdots\varphi}_{p}*\underbrace{\hat{\varphi}*\cdots*\hat{\varphi}}_{q})=(T\varphi)^p(\overline{T\varphi})^q$ has the same support as $T\varphi$ we get also $\int_{U_x}(T(\varphi)^p(\overline{T\varphi})^q g(\xi)\,d\xi^2\cdots d\xi^n=(E\varphi(x))^p(\overline{E\varphi(x)})^q$; here U_x is the set of all points ξ in U where $\xi^1=r(\xi)=x$.

Now we shall prove that $|T\varphi(\xi)|$ is equal to $|E\varphi(x)|$ or 0 in U_x. If at some point ξ_0 in U_x $|T\varphi(\xi_0)|>|E\varphi(x)|$

then
$$|E\varphi(x)|^2=\int_{U_x}|T\varphi|^2\,d\xi'>0\ (d\xi'=g(\xi)\,d\xi^2\cdots d\xi^n)$$
and
$$|T\varphi(\xi)|>|E\varphi(x)|(1+\varepsilon)$$
in some neighbourhood V of ξ_0 in U_x with some positive number ε. Therefore $1=\int_{U_x}|T\varphi|^{2p}\,d\xi'\div|E\varphi(x)|^{2p}>(1+\varepsilon)^{2p}$ volume (V) for every natural number p. But it is impossible and
$$|T\varphi(\xi)\leqq|E\varphi(x)| \text{ in } U_x.$$
If
$$0<|T\varphi(\xi_0)|<|E\varphi(x)|$$
then
$$1=\frac{\int_{U_x}|T\varphi|^{2(p+1)}\,d\xi'}{|E\varphi(x)|^{2(p+1)}}<\frac{\int_{U_x}|T\varphi|^{2p}\,d\xi'}{|E\varphi(x)|^{2p}}=1.$$
So it must be $|T\varphi(\xi)|=0$ or $|E\varphi(x)|$ in U_x. But U_x is connected and therefore
$$|T\varphi(\xi)|=|E\varphi(x)| \quad \text{in } U_x.$$

If ξ_1 and ξ_2 are different points in U_x we can take as φ, by Hypothesis II_1, such a function that $T\varphi(\xi_1)\neq 0$ and the support of $T\varphi$ is contained in U but does not contain ξ_2. On the other hand by the above result we have $|T\varphi(\xi_1)|=|T\varphi(\xi_2)|$. This contradiction shows that U_x consists of a single point and $\mathfrak{R}$ is one-dimensional, moreover $r:\xi\rightarrow r(\xi)$ is a locally bijective mapping. In other words with a suitable orientation $r(\xi)$ is monotonically increasing at every point ξ, therefore r is a one to one mapping from $\mathfrak{R}$ into $\boldsymbol{R}$.

Now wo have
$$\int_{\boldsymbol{R}}\exp(-2\pi\, ixa)\,E\varphi(x)dx=\varphi(a)$$
$$=\int_U T\varphi_a(\xi)\,d\xi=\int_U\exp(-2\pi\, ir(\xi)a)T\varphi(\xi)d\xi$$
$$=\int_{r(U)}\exp(-2\pi\, ixa)T\varphi(r^{-1}(x))g(x)dx.$$
here $d\xi=g(x)dx$ for $x=r(\xi)$. Therefore $T\varphi(r^{-1}(x))g(x)=E\varphi(x)$. If we apply this formula to $\varphi*\cdots*\varphi$ we have
$$(T\varphi(r^{-1}(x)))^m g(x)=(E\varphi(x))^m$$

for $m=1,2,3,\cdots$. From this $g(x)=1$ in $r(\mathfrak{R})$ and $T\varphi(\xi)=E\varphi(r(\xi))$ for any function in $\mathfrak{P}$ with a small compact support.

Let $T\varphi(\xi_0)$ differ from 0 and ψ be any function in $\mathfrak{P}$. The support of $T(\varphi*\psi)$ is contained in the support of $T\varphi$ and we can conclude $T(\varphi*\psi)(\xi_0)=E(\varphi*\psi)(r(\xi_0))$ or $T\varphi(\xi_0)T\psi(\xi_0)=E\varphi(r(\xi_0))E\psi(r(\xi_0))$. So we get $T\psi(\xi_0)=E\psi(r(\xi_0))$.

Thus, with Lemma 3, we have completed the proof of the theorem.

4. We investigate the case $\mathfrak{R}=\boldsymbol{R}$. By the previous result $g(x)=1$ we have
$$dr(\xi)=d\xi.$$

Proposition. *If* $\mathfrak{R}=\boldsymbol{R}$ *then*
$$r(\xi)=\xi+c$$
with a constant c. *Under Hypothesis V* c *is an integer.*

References

[1] S. Bochner and Chandrasekharan: Fourier Transform, Ann. Math. Studies, 19, Princeton (1949).

[2] K. Iwasaki: Some characterizations of Fourier transforms, Proc. Japan. Acad., **35**, no. 8, 423-426 (1959).

[3] L. Schwartz: Théorie des Distribution, Herman, Paris (1950).