139. *Some Results in Lebesgue Geometry of Curves*

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1. **Borel-rectifiability of a curve on a set.** We shall resume the study of measure-theoretic properties of parametric curves set forth in our recent notes [4] and [5]. A curve φ, situated in a Euclidean space $\boldsymbol{R}^m$ of any dimension, will be said to be *Borel-rectifiable* (or *B-rectifiable*, for short) on a set E of real numbers, when and only when E admits an expression as the join of a sequence of sets which, if E is nonvoid, are relatively Borel with respect to E and on each of which φ is rectifiable. In other words, E can be covered by a sequence of Borel sets (in the absolute sense) on each of whose intersections with E the curve φ is rectifiable. As may be immediately seen, *this is certainly the case when φ is countably rectifiable on E and at the same time continuous on E.*

We are now in a position to generalize the theorem of [5]§3 to the following form, the proof being the same as before.

THEOREM. *For each function $f(t)$ which is Borel-rectifiable on a Borel set E, the multiplicity $N(f;x;E)$ is a measurable function of x and its integral over the real line coincides with $\Xi(f;E)$ and with $\Gamma(f;E)$.*

Moreover, an inspection of part 2) of the proof for the theorem of [5]§2 leads readily to the following extension of that theorem.

THEOREM. *If a curve φ is Borel-rectifiable on a set E, then $\Xi(\varphi;E)$ coincides with $\Gamma(\varphi;E)$.*

Let us make a few remarks. The function $f(t)$, defined to be 0 or 1 according as t is rational or irrational, gives an example to the last theorem when we consider the unit interval $I=[0,1]$ for instance. Since $f(t)$ is neither continuous on I nor rectifiable (i.e. of bounded variation) on I, this case is not covered by the theorem of [5]§2. On the other hand we cannot decide at present whether B-rectifiability may be replaced in our result by countable rectifiability or by a still weaker condition. But we can at least assert that B-rectifiability of φ on E is not always necessary for the coincidence of $\Xi(\varphi;E)$ and $\Gamma(\varphi;E)$.

In fact, put $I=[0,1]$ as above and choose a non-measurable set $A\subset I$. Then the characteristic function of the set A, for which we shall write $g(t)$, is obviously countably rectifiable (that is, VBG) on I and we find immediately that $\Xi(g;I)=\Gamma(g;I)=0$. We proceed to verify that g is not B-rectifiable on I. Supposing that the contrary were true, let us express I, as we may, as the join of an infinite sequence of nonvoid Borel sets $B_1, B_2, \cdots$ on each of which g is VB. For each $n=1, 2, \cdots$ we denote by C_n the set of the points of B_n at which the subfunction $(g; B_n)$, i.e. the restriction of g to the set B_n, is discontinuous. We observe in passing that every point of C_n must then be a point of accumulation for B_n. We shall now show that, among the sets $C_1, C_2, \cdots$ thus constructed, there exists at least one which is infinite. Indeed, if this were false, $B_n - C_n$ would be a Borel set for each n. But evidently g is continuous on $B_n - C_n$. Consequently $AB_n - C_n$, which consists of all the points t of $B_n - C_n$ such that $g(t)=1$, must be a Borel set. Since $AB_n = (AB_n - C_n) \smile (AC_n)$ for each n and since further $A = AB_1 \smile AB_2 \smile \cdots$, it follows that A is a Borel set. This contradicts the definition of A.

We can thus choose a natural number p such that C_p is infinite. Consider in C_p any finite sequence $t_1, \cdots, t_k$ of k distinct points. Denoting for $i=1, 2, \cdots, k$ by W_i the oscillation of the function $(g; B_p)$ at the point t_i, we see at once that $W_i = 1$. This, in combination with the evident relation $L(g; B_p) \geqq W_1 + \cdots + W_k$, shows that $L(g; B_p) \geqq k$. Making $k \to +\infty$ we deduce $L(g; B_p) = +\infty$, which is incompatible with the definition of the sequence $B_1, B_2, \cdots$ and proves that, as we have asserted, g is not B-rectifiable on I.

2. **Another definition of reduced measure-length.** Given a curve φ and a set E, consider any curve ψ which coincides with φ on E. The infimum of the measure-length $L_*(\psi; E)$ for all such curves ψ will be called *essential measure-length* of φ over E and written $L_0(\varphi; E)$. We observe that $L_0(\varphi; E)$, thus defined, depends solely on the behaviour of φ within the set E. Now the reduced measure-length $\Xi(\varphi; E)$, introduced in [4]§2, can be given a second definition in terms of essential measure-length. This we shall state in the form of a theorem as follows.

THEOREM. *Given φ and E as above, represent E arbitrarily as the join of a sequence Δ (finite or not) of its subsets. Then $\Xi(\varphi; E)$ coincides with the infimum of $L_0(\varphi; \Delta)$ for all Δ.*

PROOF. Let ψ have the same meaning as above. The lemma of [4]§2 then implies $\Xi(\varphi; E) = \Xi(\psi; E) \leqq L_*(\psi; E)$, and it follows at once that $\Xi(\varphi; E) \leqq L_0(\varphi; E)$. Here the set E may plainly be replaced by any other set. Therefore $\Xi(\varphi; \Delta) \leqq L_0(\varphi; \Delta)$ for each sequence Δ of the assertion. On the other hand we have $\Xi(\varphi; E) \leqq \Xi(\varphi; \Delta)$, since the reduced measure-length is an outer Carathéodory measure. Consequently $\Xi(\varphi; E) \leqq L_0(\varphi; \Delta)$ and so, denoting for the moment by $\Xi_0(\varphi; E)$ the infimum of $L_0(\varphi; \Delta)$ for all Δ, we get the inequality $\Xi(\varphi; E) \leqq \Xi_0(\varphi; E)$.

We have to derive further the converse inequality. By definition, $\Xi(\varphi; E)$ is the infimum of $L(\varphi; \Delta)$ for all Δ, so that it is sufficient to verify that $\Xi_0(\varphi; E) \leqq L(\varphi; \Delta)$ for each Δ. But we easily infer from the definition of Ξ_0 that $\Xi_0(\varphi; E) \leqq \Xi_0(\varphi; \Delta)$. Our theorem will therefore be established if we show that $\Xi_0(\varphi; X) \leqq L(\varphi; X)$ for each given set X, where we may and do assume the right-hand side finite. In virtue of Lemma (4.1) stated on p. 221 of Saks [6], we may then suppose further that the curve φ is rectifiable (on the whole $\boldsymbol{R}$).

This being so, let K denote the set of all the points of discontinuity for φ. Then K must be countable since φ is rectifiable. Accordingly $\Xi_0(\varphi; KX)$ vanishes by definition, and therefore, writing for short $Y = X - K$, we find immediately

$$\Xi_0(\varphi; X) \leqq \Xi_0(\varphi; Y) + \Xi_0(\varphi; KX) = \Xi_0(\varphi; Y) \leqq L_0(\varphi; Y).$$

On the other hand $L_0(\varphi; Y) \leqq L_*(\varphi; Y) \leqq L(\varphi; Y) \leqq L(\varphi; X)$ on account of the theorem of [4]§4. Hence $\Xi_0(\varphi; X) \leqq L(\varphi; X)$, which completes the proof.

3. **Unit-spheric curves.** In the rest of this note the space $\boldsymbol{R}^m$ will be expressly assumed to be at least 2-dimensional. Suppose that $\gamma(t)$ is a *unit-spheric curve* (or simply a *spheric curve*) in $\boldsymbol{R}^m$, i.e. let $|\gamma(t)| = 1$ for every $t \in \boldsymbol{R}$. The *spheric length* and the *spheric measure-length* of γ on a set E, we define as in [1]§39 and in [2]§5 respectively. As before they will be written $\Lambda(\gamma; E)$ and $\Lambda_*(\gamma; E)$, where the reference to γ may be omitted when this causes no ambiguity. We are going to prove a theorem which will give, in terms of spheric length, a third definition to the reduced measure-length $\Xi(\gamma; E)$ induced by γ. Before doing so, however, we must establish the following auxiliary result.

LEMMA. *If a spheric curve γ is rectifiable on a set E, there exists a rectifiable spheric curve which coincides with γ at all points of E.*

REMARK. As we observed in [1]§40, a spheric curve is rectifiable on a set iff it is spherically rectifiable on the same set.

PROOF. Supposing E nonvoid as we may, consider its closure $\bar{E}$. We construct on $\bar{E}$ a spheric curve $\upsilon(t)$ as follows. For each point t_0 of E we set simply $\upsilon(t_0) = \gamma(t_0)$. When on the other hand $t_0 \in \bar{E} - E$, we distinguish two cases according as t_0 is a left-hand point of accumulation for E, or not. In the former case $\gamma(t)$ tends, by hypothesis, to a definite limit as t tends to t_0 in an increasing manner by values belonging to E, and we define $\upsilon(t_0)$ equal to this limit. In the latter case t_0 must be a right-hand point of accumulation for E, and we define $\upsilon(t_0)$ correspondingly in an obvious way. We then see immediately that υ is a spheric curve on $\bar{E}$ and that

$\Lambda(\upsilon; \bar{E})=\Lambda(\gamma; E)<+\infty$. This allows us to assume from the first that E is a nonvoid closed set. Of course, we may further restrict to the case $E\neq\boldsymbol{R}$.

To construct a spheric curve ξ which conforms to the assertion, we put in the first place $\xi(t)=\gamma(t)$ for each $t\in E$, as required by the assertion. We then extend the definition of $\xi(t)$ to the remaining points as follows. Let I denote generically an interval contiguous to E, that is to say, the closure of a connected component of the nonvoid open set $\boldsymbol{R}-E$. We have two cases to distinguish according as I is a finite or infinite interval. In the second case, I plainly has one of the two forms $[p, +\infty)$ and $(-\infty, p]$, and noting that $p\in E$, we put simply $\xi(t)=\xi(p)$ for all points $t\neq p$ of I, so that $\xi(t)$ is constant on I.

Passing to the first case let us write $I=[a, b]$, where $a\in E$ and $b\in E$. If now $\gamma(a)+\gamma(b)\neq 0$, we put $\sigma(t)=(1-\lambda)\gamma(a)+\lambda\gamma(b)$ for each point t of the open interval (a, b), the number λ being determined by the equation $t=(1-\lambda)a+\lambda b$. Then evidently $\sigma(t)\neq 0$, and we define $\xi(t)$ to be the direction of the vector $\sigma(t)$, i.e. we set $\xi(t)=|\sigma(t)|^{-1}\sigma(t)$. If on the other hand $\gamma(a)+\gamma(b)=0$, we denote by c the middle point of I and, in order to define $\xi(t)$ on (a, b), we first determine $\xi(c)$ to be any unit-vector of the space $\boldsymbol{R}^m$ different from both $\gamma(a)$ and $\gamma(b)$. Then neither $\gamma(a)+\xi(c)$ nor $\gamma(b)+\xi(c)$ vanishes, and so we can proceed in the same way as above to define $\xi(t)$ on each of the two intervals (a, c) and (c, b).

The spheric curve $\xi(t)$, thus defined over the real line and coinciding with $\gamma(t)$ on E, must be rectifiable. In fact, we can even prove the stronger relation $\Lambda(\xi; \boldsymbol{R})=\Lambda(\gamma; E)$. The verification is not difficult and may be left out.

THEOREM. *Given a spheric curve γ and a set E, let Δ denote any sequence consisting of subsets of E and covering E. Then $\Xi(\gamma; E)$ equals the infimum of $\Lambda(\gamma; \Delta)$ for all Δ.*

PROOF. Let $\Lambda_0(E)$ stand for the infimum under consideration. We need only derive $\Lambda_0(E)\leqq\Xi(E)$, for the converse inequality is an immediate consequence of the relation $\Lambda(X)\geqq L(X)$ which holds for every set X. We inspect the proof of the theorem of the foregoing § and find at once that the second paragraph of that proof remains valid if we replace there the letters φ and Ξ_0 throughout by γ and Λ_0 respectively and if, further, we use the above lemma instead of Lemma (4.1) on p. 221 of Saks [6]. It is thus enough to establish $\Lambda_0(X)\leqq L(X)$ for each set X, the spheric curve γ being now assumed rectifiable (over the whole $\boldsymbol{R}$).

With the help of the technique that was used in the proof of the above lemma in order to define the curve $\xi(t)$, we may then repeat for γ and E an argument essentially the same as that made in the proof of the theorem of [4]§4 for the construction of the curve $\omega(u)$. This enables us to suppose further that γ is a continuous curve.

Now the theorem of [4]§4 ensures $L_*(X) \leqq L(X)$, while the lemma of [3]§5 gives $\Lambda_*(B) = L_*(B)$ for every Borel set B. Since $\Lambda_*(X)$ is the infimum of $\Lambda_*(B)$ for all $B \supset X$ and similarly for the measure-length, it follows that $\Lambda_*(X) = L_*(X)$. Moreover we easily prove $\Lambda_0(X) \leqq \Lambda_*(X)$, as for the lemma of [4]§2. We thus obtain $\Lambda_0(X) \leqq L(X)$, completing the proof.

4. Reduced measure-bend of a curve over a set. For any curve $\varphi(t)$ situated in $\boldsymbol{R}^m$, where $m \geqq 2$ as was remarked in the foregoing §, we may define as in [1]§28 the *bend* of φ over a set E. We shall denote it by $\Omega(\varphi; E)$ as before. By the *reduced measure-bend* of φ over E, written $\Upsilon(\varphi; E)$, we shall now understand the infimum of the sum $\Omega(\varphi; \Delta)$, where Δ is an arbitrary sequence of subsets of E which covers E. When there is no fear of confusion, we may write $\Omega(E)$ and $\Upsilon(E)$ for these two quantities. It should be noted that we have not assumed the lightness of the curve φ in the above.

LEMMA. *Given φ and E as above, let $\Theta = \langle I_1, I_2, \cdots \rangle$ be an arbitrary non-overlapping sequence of intervals and let us write for short $\Theta E = \langle I_1 E, I_2 E, \cdots \rangle$. Then $\Omega(\Theta E) \leqq \Omega(E)$.*

PROOF. This extension of the proposition of [1]§31 may be established in almost the same way as for that proposition.

THEOREM. *The reduced measure-bend $\Upsilon(\varphi; E)$, considered as a function of the set E, is an outer measure of Carathéodory which vanishes whenever E is a countable set.*

PROOF. Clearly we have $\Upsilon(E) = 0$ for countable E. We must verify further the following three conditions: (i) $\Upsilon(X) \leqq \Upsilon(Y)$ whenever $X \subset Y$; (ii) $\Upsilon([\Delta]) \leqq \Upsilon(\Delta)$ for any sequence Δ of sets; (iii) $\Upsilon(X \smile Y) \geqq \Upsilon(X) + \Upsilon(Y)$ for any pair of nonvoid sets X and Y with positive distance. Conditions (i) and (ii) being obvious, we may confine ourselves to (iii). By hypothesis there is a disjoint pair of open sets A and B containing X and Y respectively. Let Φ be a sequence consisting of all the connected components of A, and let Ψ be defined similarly for B. Then Φ as well as Ψ is plainly a disjoint sequence of endless intervals, no element of Φ intersecting any element of Ψ. Accordingly, by our lemma, $\Omega(N) \geqq \Omega(\Phi N) + \Omega(\Psi N)$ for each set $N \subset X \smile Y$. If, therefore, we express $X \smile Y$ arbitrarily as the join of a sequence Θ of its subsets, then $\Omega(\Theta) \geqq \Upsilon(X) + \Upsilon(Y)$. This implies condition (iii) and completes the proof.

5. Outer bend of point sets. We define firstly a set-function $\omega(X)$ for finite sets X in $\boldsymbol{R}^m$ (where $m \geqq 2$) as follows. When X

consists of at most two points, we set $\omega(X)=0$. Otherwise arrange all the points of X in any distinct sequence $x_0, x_1, \cdots, x_n$ and write $p_i=x_i-x_{i-1}$ for $i=1,\cdots,n$. We understand by $\omega(X)$ the minimum, for all such sequences, of the angle-sum $p_1\diamond p_2+\cdots+p_{n-1}\diamond p_n$. We now extend ω to infinite sets $Y\subset \boldsymbol{R}^m$. Let namely $\omega(Y)$ mean for each Y the supremum of $\omega(X)$ for all finite subsets X of Y. Thus defined for all sets in $\boldsymbol{R}^m$ the function ω is monotone non-decreasing, as we readily see with the aid of [1]§25.

Given a set $M\subset \boldsymbol{R}^m$ and a positive number ε, let us consider the infimum of the sum $\omega(\Theta)$, where Θ is an arbitrary sequence of subsets of M whose join is M and whose diameters are less than ε. When $\varepsilon\to 0$, this infimum tends in a non-decreasing manner to a limit, which will be denoted by $\omega_0(M)$ and termed *outer bend* of M (cf. the definition of outer length stated on p. 54 of Saks [6]). In view of monotonity of ω we verify at once that *the outer bend is an outer Carathédory measure in* $\boldsymbol{R}^m$ *vanishing for countable sets.*

Needless to say, the notion of reduced measure-bend introduced in the preceding section is an analogue, in bend theory, of the reduced measure-length. But there also exists in bend theory a notion which is analogous to the Hausdorff measure-length and which we propose to call *Hausdorff measure-bend.* To obtain the latter we need merely replace, in the definition of Hausdorff measure-length, the diameters of point sets in $\boldsymbol{R}^m$ by their ω-values. The definition in full, as well as some basic properties, of this new quantity will be given elsewhere in the near future.

References

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