10. On the Reduced Measure-Bend of Curves

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1. A property of B-measurable curves. Continuing our recent note [4], we shall commence the present study with the following result, whose proof is based on an argument essentially the same as that of the proof for Theorem (4.3) stated on p. 113 of Saks [5] and concerning the measurability of Dini derivates. The underlying space $\boldsymbol{R}^m$ will be assumed at least 2-dimensional unless specified otherwise.

THEOREM. *Given a B-measurable curve φ and a B-measurable spheric curve γ (both situated in $\boldsymbol{R}^m$), let K be the set of the points t at which $\gamma(t)$ is a right-hand derived direction for φ. Then K is a Borel set.*

REMARK. Needless to say, a curve is called B-measurable iff all its coordinate functions are B-measurable.

PROOF. For each pair p, q of natural numbers let us define a point set $K(p, q)\subset\boldsymbol{R}$ as follows: a point t of $\boldsymbol{R}$ belongs to $K(p, q)$ when and only when there exists an x such that

$$t+q^{-1}<x<t+p^{-1}, \quad |\varphi(x)-\varphi(t)|>q^{-1}, \quad \gamma(t)\diamond[\varphi(x)-\varphi(t)]<p^{-1}.$$

We see at once that $K(p, q)$ is descending and ascending in p and q respectively and that the set K of the assertion may be written

$$K=\lim_p \lim_q K(2p, q)=\lim_p \lim_q K(p, 2q).$$

Since φ is B-measurable, we can easily associate with each $n=1, 2,\cdots$ a B-measurable curve $\varphi_n(t)$ such that the image $\varphi_n[\boldsymbol{R}]$ is countable and that $|\varphi_n(t)-\varphi(t)|<n^{-1}$ for every $t\in\boldsymbol{R}$. Similarly there is a B-measurable spheric curve $\gamma_n(t)$ such that $\gamma_n[\boldsymbol{R}]$ is countable and $|\gamma_n(t)-\gamma(t)|<n^{-1}$ for every t. So that the two sequences of curves, $\langle\varphi_1(t), \varphi_2(t),\cdots\rangle$ and $\langle\gamma_1(t), \gamma_2(t),\cdots\rangle$, tend respectively to $\varphi(t)$ and $\gamma(t)$ uniformly on the real line. Let us now replace, in the above definition of the set $K(p, q)$, the curves φ and γ by φ_n and γ_n respectively and write $K_n(p, q)$ for the resulting set. We then denote for short the two limits

$$\operatorname*{lim\text{-}inf}_n K_n(p, q) \quad\text{and}\quad \operatorname*{lim\text{-}sup}_n K_n(p, q)$$

by $U(p, q)$ and $V(p, q)$ respectively and find readily that

$$K(2p, q)\subset U(2p, q)\subset V(2p, q)\subset K(p, 2q).$$

From this we deduce, by what has already been proved, that

$$K\subset\lim_p \lim_q U(2p, q)\subset\lim_p \lim_q V(2p, q)\subset K.$$

Consequently we may replace in this relation the signs of inclusion throughout by those of equality. It follows that our theorem will be established if we verify each $K_n(p, q)$ to be a Borel set.

Now the definitions of the curves φ_n and γ_n plainly imply that there exists for each n a decomposition of the real line into a sequence (countable, of course) of nonvoid Borel sets B on each of which both φ_n and γ_n are constant. Then the intersection of the set $K_n(p, q)$ with each B, which is immediately seen to be open in B, must be a Borel set. Hence so must also be $K_n(p, q)$ itself, and this completes the proof as observed above.

2. A relation involving bend, measure-bend, and reduced measure-bend. Let us premise an auxiliary inequality which may be established in the same way as for the lemma of [2]§2.

LEMMA. *We have $\Upsilon(\varphi; E) \leqq \Omega_*(\varphi; E)$ for any curve φ and any set E, where $\Upsilon(\varphi; E)$ is the reduced measure-bend of φ on E defined in* [3]§4 *and the measure-bend $\Omega_*(\varphi; E)$ is interpreted in its extended sense as remarked in* [4]§3.

It was proved in [2]§4 that if a locally rectifiable curve ψ is continuous at all points of a set E, then $\Xi(\psi; E) = L_*(\psi; E) \leqq L(\psi; E)$. (Here the Euclidean space in which the curve ψ lies may exceptionally be of any dimension. All the other curves considered in this note will be situated in $\boldsymbol{R}^m$, where $m \geqq 2$.) This result has now the following analogue in bend theory, the derivation of which is the object of the present section.

THEOREM. *Given a curve φ and a set E, suppose that $\Omega_*(\varphi; X)$ vanishes for every countable set $X \subset E$. Then*

$$\Upsilon(\varphi; E) = \Omega_*(\varphi; E) \leqq \Omega(\varphi; E).$$

PROOF. 1) The equality of the assertion follows at once from the inequality. For, whenever a sequence Δ of subsets of E covers E, we have $\Omega_*(\varphi; E) \leqq \Omega_*(\varphi; \Delta) \leqq \Omega(\varphi; \Delta)$ provided our inequality is true. By definition of reduced measure-bend this yields readily $\Omega_*(\varphi; E) \leqq \Upsilon(\varphi; E)$, which combined with the above lemma leads to the equality.

2) The inequality. We observe first that if an endless interval is expressed as the join of a finite sequence Θ of n endless intervals, then $\Omega(\varphi; [\Theta])$ cannot exceed $\Omega(\varphi; \Theta) + (n-1)\pi$; indeed this is easily proved when $n=2$, and then the general case follows directly by induction. On inspecting now the proof for part (i) of the theorem of [4]§3 it is immediately found in view of what has just been said that, in order to derive the inequality of our theorem, we may add without loss of generality the hypothesis that φ is straightenable, i.e. that $\Omega(\varphi; \boldsymbol{R}) < +\infty$. Moreover the set E may be supposed bounded, since $\Omega_*(\varphi; [-n, n]E) \to \Omega_*(\varphi; E)$ as n tends to infinity by positive

integral values. These two assumptions will be kept throughout the rest of the proof. Our argument will conveniently be divided into four parts in which φ is further supposed to be respectively (a) continuous and light, (b) continuous, (c) locally rectifiable, and finally (d) not locally rectifiable. The latter three cases will be reduced each to the preceding one.

Consider the first case. Since φ is straightenable, its measure-bend is a bounded set-function. Hence, if C denotes the set of all the points t at which $\Omega_*(\varphi; \{t\})=0$, we see that $\boldsymbol{R}-C$ is countable and that C is therefore a Borel set. For any set $Y\subset\boldsymbol{R}$, on the other hand, $\Omega_*(\varphi; Y)$ is plainly the infimum of $\Omega_*(\varphi; G)$ for all open sets $G\supset Y$. From these facts we can draw two direct conclusions. Firstly, there exists a bounded Borel set $B\supset E$ such that $\Omega_*(\varphi; B)=\Omega_*(\varphi; E)$ and that $\Omega_*(\varphi; X)=0$ whenever X is a countable subset of B. Here we may of course subject B to the further condition $B\subset\overline{E}$, and then it is evident by continuity of φ that $\Omega(\varphi; B)=\Omega(\varphi; E)$. Secondly, $\Omega_*(\varphi; B)$ is the supremum of $\Omega_*(\varphi; F)$ for all closed sets $F\subset B$. It thus follows that the set E may be assumed not only bounded, but also closed and uncountable. Let us remark explicitly, in passing, that the closedness and uncountability of E will only be utilized in dealing with the present case (a), and not for the remaining three cases. It is now possible to construct from φ a new curve ψ in precisely the same way as in the proof for the theorem of [2]§4 quoted in the above. Thus ψ coincides on E with φ and is linear on each interval contiguous to E.

This being so, denote by $\mathfrak{M}$ the family of all the closed intervals I for each of which $E-I$ is a countable set. Such intervals certainly exist since E is bounded. Writing I_0 for the intersection of the family $\mathfrak{M}$, we find that $E-I_0$ is countable. Indeed, $\boldsymbol{R}-I_0$ is the union of all the open sets $\boldsymbol{R}-I$, so that there exists in $\mathfrak{M}$, by the Lindelöf covering theorem, an infinite sequence of intervals $J_n(n=1, 2,\cdots)$ whose complements already cover $\boldsymbol{R}-I_0$. Then $E-I_0$, which is the join of the countable sets $E-J_n$, must itself be countable. This, coupled with the uncountability of E, shows that I_0 is an infinite set. Since moreover I_0 is by definition bounded, closed, and convex, it follows finally that I_0 must be a closed interval, say $[a_0, b_0]$. Accordingly, if we write $K=(a_0, b_0)$ for short, $E-K$ is countable. It is further evident that the points a_0 and b_0 are respectively a right-hand and a left-hand point of accumulation for E. Recalling now the construction of the curve ψ, we therefore find without difficulty that

$$\Omega_*(\psi; K)\leqq\Omega(\psi; K)=\Omega(\psi; EK)\leqq\Omega(\psi; E)=\Omega(\varphi; E).$$

On the other hand $\Omega_*(\varphi; E-K)$ vanishes by hypothesis and by countability of $E-K$, so that $\Omega_*(\varphi; E)=\Omega_*(\varphi; EK)$. But part (ii) of the theorem of [4]§3 ensures $\Omega_*(\varphi; EK)\leqq\Omega_*(\psi; EK)$. The results obtained in the above imply together that

$$\Omega_*(\varphi; E)\leqq\Omega_*(\psi; EK)\leqq\Omega_*(\psi; K)\leqq\Omega(\varphi; E),$$

which proves the required inequality under the assumption (a).

We shall now turn to the consideration of case (b). The set E being bounded, we may suppose that there is an open interval $(\alpha, \beta)\supset E$ such that the curve φ is rectifiable on neither of the intervals $(-\infty, \alpha)$ and $(\beta, +\infty)$. Noting that φ is locally rectifiable in conformity with [1]§64, let $s(t)$ be a length-function for φ, i.e. any real function such that $s(t')-s(t)=L(\varphi; [t, t'])$ whenever $t<t'$. Evidently $s(t)$ is then a non-decreasing continuous function with the whole real line for its range. With the help of this function we now construct from φ a new curve $\xi(u)$ by putting $\xi(u)=\varphi(t)$ for each $u\in\boldsymbol{R}$, where t is any point fulfilling $s(t)=u$. This defines ξ uniquely since we have $\varphi(t')=\varphi(t)$ whenever $s(t')=s(t)$. We see immediately that ξ is a light continuous curve and that $\Omega(\xi; s[X])=\Omega(\varphi; X)$ for every set $X\subset\boldsymbol{R}$ (in particular, therefore, ξ is straightenable). Thus it is sufficient to deduce $\Omega_*(\varphi; E)\leqq\Omega(\xi; s[E])$.

This being so, let A be an interval of constancy for φ, i.e. a maximal closed interval on which φ is constant, and consider any closed interval J adjacent to A. On account of [1]§35, we find that $\Omega(\varphi; J)\to 0$ as $|J|\to 0$. From this we infer at once that if I is any open interval containing an extremity of A, then $\Omega(\varphi; I)\to 0$ as $|I|\to 0$. This, combined with the definition of measure-bend, requires that $\Omega_*(\varphi; A)=0$. Since there is at most a countable infinity of the intervals A, it follows that $\Omega_*(\varphi; A_0)=0$ where A_0 stands for the join of all A. Writing $E_0=E-A_0$ we therefore get $\Omega_*(\varphi; E)=\Omega_*(\varphi; E_0)$. It is further easy to see that if $t_0\in E_0$ and $u_0=s(t_0)$, then $\Omega_*(\xi; \{u_0\}) =\Omega_*(\varphi; \{t_0\})=0$. Indeed, this is an immediate consequence of the fact that whenever I is an open interval containing t_0, its image $s[I]$ has u_0 for an interior point and $\Omega(\xi; s[I])$ equals $\Omega(\varphi; I)$. Writing for short $M=s[E_0]$ and recalling the lightness and continuity of ξ, it follows now from what we have already proved that $\Omega_*(\xi; M) \leqq\Omega(\xi; M)$. To derive the required relation $\Omega_*(\varphi; E)\leqq\Omega(\xi; s[E])$, it is therefore enough, in view of $\Omega_*(\varphi; E)=\Omega_*(\varphi; E_0)$ established above, to verify that $\Omega_*(\varphi; E_0)\leqq\Omega_*(\xi; M)$.

For this purpose, suppose M nonvoid together with E_0 and consider an arbitrary sequence Δ of endless intervals covering M, so that $\Omega_*(\xi; M)$ is by definition the infimum of $\Omega(\xi; \Delta)$ for all choices of Δ. Then the inverse image $I'=s^{-1}[I]$ of each interval I in Δ is likewise an endless interval in virtue of continuity of $s(t)$. Since clearly $I=s[I']$, we find further $\Omega(\xi; I)=\Omega(\varphi; I')$. If, consequently, Δ'

denotes the sequence obtained from Δ by replacing each I by I', it follows in view of the inclusion $E_0 \subset [\Delta']$ that $\Omega_*(\varphi; E_0) \leqq \Omega(\varphi; \Delta') = \Omega(\xi; \Delta)$. Since this implies $\Omega_*(\varphi; E_0) \leqq \Omega_*(\xi; M)$, we have proved $\Omega_*(\varphi; E) \leqq \Omega(\varphi; E)$ under the assumption (b).

Let us pass now to case (c). Let H be the countable set consisting of all the points of discontinuity of φ. In relation to H we can construct on $\boldsymbol{R}$ a pair of functions $p(u)$ and $q(t)$ in exactly the same way as in the proof for the theorem of [2]§4. By means of these functions we define further a continuous curve $\omega(u)$ by quite the same process as in the quoted proof. It is then obvious that $\Omega(\omega; q[X]) = \Omega(\varphi; X)$ for every set X and that $\Omega(\omega; p^{-1}[G]) = \Omega(\varphi; G)$ for every open set G. The latter relation implies in particular that ω is straightenable. If we now write $E_1 = E - H$, we get by hypothesis $\Omega_*(\varphi; E) = \Omega_*(\varphi; E_1)$. So that it suffices to show $\Omega_*(\varphi; E_1) \leqq \Omega(\varphi; E_1)$. For this purpose we observe in the first place that if $t_1 \in E_1$ and $u_1 = q(t_1)$, then $\Omega_*(\omega; \{u_1\}) = \Omega_*(\varphi; \{t_1\}) = 0$. In point of fact, the inverse image $p^{-1}[I]$ of each open interval I containing t_1 is also an open interval containing the point u_1, and we have $\Omega(\omega; p^{-1}[I]) = \Omega(\varphi; I)$ by what has just been said above; whence the result. Accordingly, by case (b) treated already, we obtain $\Omega_*(\omega; N) \leqq \Omega(\omega; N)$, where N abbreviates the bounded set $q[E_1]$. But $\Omega(\omega; N) = \Omega(\varphi; E_1)$, and so we need only examine the inequality $\Omega_*(\varphi; E_1) \leqq \Omega_*(\omega; N)$ in what follows.

Consider any open set D of real numbers. Since the curve φ is continuous at all points of E_1, we can easily attach to each point t of the intersection $p[D] \cdot E_1$ an open interval $U(t)$ such that $V(t) = p^{-1}[U(t)]$ is an open interval in D. If, therefore, S stands for the (possibly void) join of all the intervals $U(t)$, it follows that S is open and that the set $p^{-1}[S]$, which is clearly the join of all $V(t)$, lies in D. Consequently, noting the inclusion $p[D] \cdot E_1 \subset S$, we deduce in view of definition of measure-bend that

$$\Omega_*(\varphi; p[D] \cdot E_1) \leqq \Omega(\varphi; S) = \Omega(\omega; p^{-1}[S]) \leqq \Omega(\omega; D).$$

This being established, suppose N nonvoid together with E_1 and cover N by an arbitrary sequence Θ of endless intervals J. The last relation then implies that $\Omega_*(\varphi; p[J] \cdot E_1) \leqq \Omega(\omega; J)$ for each J. But the sets $p[J]$ together cover E_1, for evidently $E_1 = p[N]$. Therefore $\Omega_*(\varphi; E_1) \leqq \Omega(\omega; \Theta)$. Since $\Omega(\omega; N)$ is the infimum of $\Omega(\omega; \Theta)$, this leads finally to $\Omega_*(\varphi; E_1) \leqq \Omega_*(\omega; N)$, which implies the inequality of our theorem as already observed.

It remains to treat case (d). Let us begin by defining a set T of real numbers as follows: a point t belongs to T iff the curve φ is unbounded on every open interval containing t. Consider any point $t_0 \in T$ and choose any pair of points α and β such that $\alpha < t_0 < \beta$.

Then φ is unbounded on (α, β), so that for each positive integer n there is in (α, β) a point t_n for which the two vectors $\varphi(t_n)-\varphi(\alpha)$ and $\varphi(\beta)-\varphi(t_n)$ are $\neq 0$ and make an angle greater than $\pi-n^{-1}$. This implies that $\Omega(\varphi; [\alpha, \beta])\geqq\pi$, whence we derive $\Omega_*(\varphi; \{t_0\})\geqq\pi$ by making $\alpha\to t_0$ and $\beta\to t_0$ simultaneously. Since φ is straightenable, we conclude that T is a finite set. In consequence, $\boldsymbol{R}-T$ is the join of a finite disjoint sequence Δ of endless intervals and, since $\Omega_*(\varphi; T)$ vanishes by hypothesis, we have $\Omega_*(\varphi; E)=\Omega_*(\varphi; E\Delta)$. On the other hand it is obvious that $\Omega(\varphi; E\Delta)\leqq\Omega(\varphi; E)$. In order to prove $\Omega_*(\varphi; E)\leqq\Omega(\varphi; E)$, it therefore suffices to obtain $\Omega_*(\varphi; EI)\leqq\Omega(\varphi; EI)$ for each interval I which occurs in Δ. Since I is disjoint from T, each point t of I can be enclosed in an open interval $U(t)\subset I$ on which φ is bounded. Then φ must be rectifiable on $U(t)$ in virtue of the lemma of [4]§1. Therefore, by the covering theorem of Heine-Borel, φ is rectifiable on every closed interval in I. The relation $\Omega_*(\varphi; EI)\leqq\Omega(\varphi; EI)$ is now a direct consequence of what we have already proved for case (c). In fact the interval I, which is endless, can be transformed into the whole real line by a suitable change of parameter; but the restriction of the curve φ to I is then changed into a locally rectifiable curve of bounded bend. (The set EI might become an unbounded set, say E^*. But this does not matter at all, since then we need only consider bounded subsets of E^*.) Our theorem is thus completely proved.

References

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