24. Further Properties of Reduced Measure-Bend

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(Comm. by Z. SUETUNA, M.J.A., March 12, 1962)

1. Completion of a previous result. We shall be concerned with curves defined on the real line $\boldsymbol{R}$ and situated in $\boldsymbol{R}^m$, where we assume $m\geqq 2$ unless stated otherwise. By sets, by themselves, we shall understand subsets of $\boldsymbol{R}$. Continuing our recent note [6], let us begin with a theorem which completes part (ii) of the theorem of [5]§3.

THEOREM. *Given a curve φ and a set E, suppose that $\Omega_*(\varphi; M)$ vanishes for every countable set $M\subset E$. Then*

$$\varUpsilon(\varphi; E)=\Omega_*(\varphi; E)\leqq\Omega_*(\psi; E)$$

for each curve ψ which coincides on E with φ.

PROOF. The lemma and the theorem of [6]§2 require respectively that $\varUpsilon(\psi; E)\leqq\Omega_*(\psi; E)$ and $\varUpsilon(\varphi; E)=\Omega_*(\varphi; E)$. But our hypothesis on the curve ψ clearly implies $\varUpsilon(\varphi; E)=\varUpsilon(\psi; E)$. Hence the result.

REMARK. The above theorem has a counterpart in length theory, as follows. (The proof is not difficult and may be left to the reader.)

Given a curve φ and a set E, suppose that $L_*(\varphi; M)=0$ holds for every countable set $M\subset E$. Then $\Xi(\varphi; E)=L_*(\varphi; E)\leqq L_*(\psi; E)$ for each curve ψ which coincides on E with φ.

Here the space in which the two curves lie may exceptionally be of any dimension.

2. Another definition of reduced measure-bend. By the *essential measure-bend* of a curve φ over a set E, we shall mean the infimum of the measure-bend $\Omega_*(\psi; E)$, where ψ is any curve which coincides on E with φ. The notation $\Omega_0(\varphi; E)$ will be used for it. In terms of this quantity we shall now give a second definition to the notion of reduced measure-bend. Indeed the theorem of [4]§2 has the following analogue.

THEOREM. *Given a curve φ and a set E, represent E in any manner as the join of a sequence $\varDelta$ of subsets and write $\varUpsilon_0(\varphi; E)$ for the infimum of the sum $\Omega_0(\varphi; \varDelta)$. Then $\varUpsilon_0(\varphi; E)=\varUpsilon(\varphi; E)$.*

PROOF. On account of the lemma of [6]§2 we have in the first place $\varUpsilon(\varphi; E)=\varUpsilon(\psi; E)\leqq\Omega_*(\psi; E)$ for every curve ψ considered above. It ensues that $\varUpsilon(\varphi; E)\leqq\Omega_0(\varphi; E)$, where we observe that E may be replaced by any other set. Therefore $\varUpsilon(\varphi; E)\leqq\varUpsilon(\varphi; \varDelta)\leqq\Omega_0(\varphi; \varDelta)$ for every $\varDelta$, and from this we infer that $\varUpsilon(\varphi; E)\leqq\varUpsilon_0(\varphi; E)$. The deduction of the converse inequality will thus constitute the main part of the proof.

Clearly we need only verify the inequality $\Upsilon_0(\varphi; E)\leqq\Omega(\varphi; \Delta)$ for each Δ. Since $\Upsilon_0(\varphi; E)\leqq\Upsilon_0(\varphi; \Delta)$ as we easily find, this will follow if we show that $\Upsilon_0(\varphi; X)\leqq\Omega(\varphi; X)$ for each set X on which φ is straightenable.

Fixing such an X, let us define a set $T\subset \boldsymbol{R}$ as follows (cf. the final paragraph of [6]). A point t belongs to T iff t is a point of accumulation for X and further, given any open interval I containing t, the curve φ is unbounded on the intersection IX. Then T is a finite set. For, if not, there would exist for each natural number n a sequence of $2n+1$ points $t_1<t_2<\cdots<t_{2n+1}$ belonging to T. By definition of T we could then choose in the set X a sequence of $n+1$ points $u_0, u_1,\cdots, u_n$ such that $u_{i-1}<t_{2i}<u_i$ for $i=1, \cdots, n$. It would follow at once that if we write $K_i=[u_{i-1}, u_i]$, then $\Omega(\varphi; K_iX)\geqq\pi$ for each i. But this would imply

$$\Omega(\varphi; X)\geqq\Omega(\varphi; K_1X)+\cdots+\Omega(\varphi; K_nX)\geqq n\pi,$$

which is impossible since n is arbitrary.

Let us now decompose the complement of the finite set T into a disjoint finite sequence Θ of endless intervals. In view of the obvious inequalities $\Upsilon_0(\varphi; X)\leqq\Upsilon_0(\varphi; X\Theta)$ and $\Omega(\varphi; X\Theta)\leqq\Omega(\varphi; X)$, our theorem will be established if we show that $\Upsilon_0(\varphi; XA)\leqq\Omega(\varphi; XA)$ for each interval A occurring in Θ. Representing A as the join of a non-overlapping sequence Φ of closed intervals, we find $\Upsilon_0(\varphi; XA)\leqq\Upsilon_0(\varphi; X\Phi)$ and $\Omega(\varphi; X\Phi)\leqq\Omega(\varphi; XA)$, so that it is enough to prove $\Upsilon_0(\varphi; XJ)\leqq\Omega(\varphi; XJ)$ for each closed interval $J\subset A$. But it follows from the definition of T, with the help of the Heine-Borel covering theorem, that φ is bounded on Y, where and subsequently Y is short for XJ. Therefore, on account of the theorem of [5]§1, there is a straightenable curve ω coinciding on Y with φ. Now it only remains to ascertain that $\Upsilon_0(\omega; Y)\leqq\Omega(\omega; Y)$.

Consider the set H of all the points t at which $\Omega_*(\omega; \{t\})>0$. Then H is countable since ω is straightenable. Accordingly $\Upsilon_0(\omega; HY)$ vanishes by definition of Υ_0, and so we find, writing $Z=Y-H$ for short, that

$$\Upsilon_0(\omega; Y)=\Upsilon_0(\omega; Z)\leqq\Omega_0(\omega; Z)\leqq\Omega_*(\omega; Z).$$

On the other hand, since $\Omega_*(\omega; \{t\})$ plainly vanishes for every $t\in Z$, the theorem of [6]§2 shows that $\Omega_*(\omega; Z)\leqq\Omega(\omega; Z)\leqq\Omega(\omega; Y)$. Hence we get finally $\Upsilon_0(\omega; Y)\leqq\Omega(\omega; Y)$, and the proof is complete.

3. A property of locally straightenable curves. Only part (i) of the following theorem will be necessary for our purpose in hand. On the other hand, part (ii) extends the proposition of [1]§80 and is at the same time closely connected with that of [1]§83.

THEOREM. *Let γ be a direction curve (defined on $\boldsymbol{R}$) of a locally straightenable light curve φ. Then* (i) *at each point $t_0 \in \boldsymbol{R}$ for which $\Omega_*(\varphi; \{t_0\})$ vanishes, the cvrve φ has a tangent direction equal to $\gamma(t_0)$. Again,* (ii) *if c is any right-hand [or left-hand] point of continuity of φ, the right-hand limit $\gamma(c+)$ [or the left-hand limit $\gamma(c-)$] exists and φ possesses at c a right-hand [or left-hand] tangent direction equal to $\gamma(c+)$ [or to $\gamma(c-)$].*

REMARK. See §§44, 42, 77 of [1] for the terminology.

PROOF. *re* (i): Given any natural number n we can enclose the point t_0, by hypothesis, in the interior of a closed interval I_n such that $\Omega(\varphi; I_n) < 1/n$. The curve φ being light, we then have $\varphi(I) \neq 0$ for every closed interval $I \subset I_n$ on account of [1]§60. It therefore follows from [1]§63 that $\varphi(I_n) \diamond \varphi(I) < 1/n$ for such I. Since n is arbitrary, this implies that φ possesses at t_0 a tangent direction, which must then plainly coincide with $\gamma(t_0)$.

re (ii): To fix the ideas, suppose φ right-hand continuous at the point c. In view of §§31–32 of [1] it is easy to associate with each $n = 1, 2, \cdots$ a closed interval J_n with left-hand extremity c and such that $\Omega(\varphi; J_n) < 1/n$. As above we then find $\varphi(J) \neq 0$ and $\varphi(J) \diamond \varphi(J_n) < 1/n$ for each closed interval $J \subset J_n$. From this we draw two consequences: firstly, φ has at c a right-hand tangent direction $\varphi^R(c)$ such that $\varphi^R(c) \diamond \varphi(J_n) \leqq 1/n$ for every n; secondly, we have $\gamma(x) \diamond \varphi(J_n) \leqq 1/n$ for each n whenever x is an interior point of J_n. So that, by the triangular inequality (see [1]§22), we obtain $\gamma(x) \diamond \varphi^R(c) \leqq 2/n$ for such x. Making $n \to +\infty$, we conclude that $\gamma(c+)$ exists and coincides with $\varphi^R(c)$. This establishes part (ii).

4. Expression of measure-bend as ordinary and spheric measure-lengths. This may be stated as follows.

THEOREM. *Given a light curve φ and a set E of real numbers, let γ be any direction curve (defined on $\boldsymbol{R}$) of φ and suppose that $\Omega_*(\varphi; M)$ vanishes for every countable subset M of E. Then we have*

$$L_*(\gamma; E) = \Lambda_*(\gamma; E) = \Omega_*(\varphi; E).$$

PROOF. We have $\Lambda_*(\gamma; E) \leqq \Omega_*(\varphi; E)$ by the lemma of [5]§3 and it is obvious that $L_*(\gamma; E) \leqq \Lambda_*(\gamma; E)$. Hence our theorem will be established if we derive $L_*(\gamma; E) = \Omega_*(\varphi; E)$. Suppose E nonvoid and let $\mathfrak{M}$ denote the family of all the open intervals with rational extremities and on which the curve φ is straightenable. Noting that $\mathfrak{M}$ is a countably infinite family covering E, we arrange all the elements of $\mathfrak{M}$ in any distinct sequence $I_1, I_2, \cdots$ and write $U_n = I_1 \cup \cdots \cup I_n$ for $n = 1, 2, \cdots$. It suffices to prove $L_*(\gamma; EU_n) = \Omega_*(\varphi; EU_n)$ for each n, since these two quantities tend to the limits $L_*(\gamma; E)$ and $\Omega_*(\varphi; E)$ respectively as $n \to +\infty$. Now each U_n can clearly be decomposed into a finite disjoint sequence Δ_n of open intervals, so that $L_*(\gamma; EU_n)=L_*(\gamma; E\Delta_n)$ and $\Omega_*(\varphi; EU_n)=\Omega_*(\varphi; E\Delta_n)$. Accordingly we need only show $L_*(\gamma; EI)=\Omega_*(\varphi; EI)$ for each interval I in Δ_n. But, since φ is light, we have $\Omega(\varphi; I)=\Omega_*(\varphi; I)$ by the theorem of [2]§3, while $\Omega_*(\varphi; I)\leqq\Omega_*(\varphi; U_n)<+\infty$ by definition of U_n. So that φ must be straightenable over I.

Arguing now as in §2 we find the existence of a finite disjoint sequence Θ of open intervals such that $I-[\Theta]$ is a finite set and that φ is bounded on every closed interval contained in $[\Theta]$. Then $\Omega_*(\varphi; EI)=\Omega_*(\varphi; E\Theta)$ by our hypothesis on the set E; while we also have $L_*(\gamma; EI)=L_*(\gamma; E\Theta)$ since, by the lemma of [5]§3 and the same hypothesis, $L_*(\gamma; \{t\})$ vanishes whenever $t\in E$. Thus it only remains to examine the validity of $L_*(\gamma; EI_0)=\Omega_*(\varphi; EI_0)$ for each interval I_0 in Θ. By change of parameter, however, this amounts to proving $L_*(\gamma; E)=\Omega_*(\varphi; E)$ under the additional assumption that φ is straightenable and locally bounded. Then φ is locally rectifiable in virtue of the lemma of [5]§1.

This being so, consider the set H of all the points of discontinuity for φ. Since H is countable, there exists by [1]§94 a non-decreasing continuous function $p(u)$ mapping $\boldsymbol{R}$ onto itself and such that the inverse image $p^{-1}(t)$ of a point $t\in\boldsymbol{R}$ under p is non-degenerate and hence a closed interval when and only when $t\in H$. We now construct on $\boldsymbol{R}$ a strictly increasing function $q(t)$ as follows. If $t\in\boldsymbol{R}-H$, we understand by $q(t)$ the point $p^{-1}(t)$. If on the other hand $t\in H$, we write $p^{-1}(t)=[a, b]$ and $c=(a+b)/2$, and we define $q(t)$ to be the point a, b, or c according as

$$\varphi(t-)=\varphi(t),\ \varphi(t+)=\varphi(t),\ \text{or}\ \varphi(t-)\neq\varphi(t)\neq\varphi(t+)$$

respectively. (Note that the first two of these three cases exclude each other on account of discontinuity of φ at t.)

Without difficulty we then can determine uniquely a light continuous curve $\omega(u)$ by the requirements that $\omega(u)=\varphi(p(u))$ for each $u\in q[\boldsymbol{R}]$ and further that $\omega(u)$ be a linear function of u on each interval of arbitrary type disjoint from $q[\boldsymbol{R}]$. As may be verified at once, we then have $\Omega(\omega; p^{-1}[D])=\Omega(\varphi; D)$ for any open set $D\subset\boldsymbol{R}$, and it follows in particular that ω is straightenable. It also follows that $\Omega_*(\omega; \{u_0\})=0$ whenever $p(u_0)\in E$. Indeed, if K is an open interval containing the point $p(u_0)$, then $p^{-1}[K]$ must be an open set containing u_0, and so

$$\Omega_*(\omega; \{u_0\})\leqq\Omega(\omega; p^{-1}[K])=\Omega(\varphi; K).$$

But $\Omega_*(\varphi; \{p(u_0)\})=0$ and therefore we can make $\Omega(\varphi; K)$ arbitrarily small by choosing K suitably. Hence the result.

As a consequence we find that $\Omega_*(\omega; N)=0$ for every countable set $N\subset q[E]$. This, combined with the theorem of [6]§2, shows that $\Omega_*(\omega; X)=\Gamma(\omega; X)$ whenever $X\subset q[E]$. In particular we may take for X the set $Q=q[E-H]$. Recalling now the definition of the curve ω and noting that q is a strictly increasing function, we get immediately $\Upsilon(\omega; Q)=\Upsilon(\varphi; E_0)$, where E_0 is short for E–H. But the theorem of [6]§2, applied this time to the curve φ, yields us the equality $\Upsilon(\varphi; E_0)=\Omega_*(\varphi; E_0)$. We thus derive $\Omega_*(\varphi; E)=\Omega_*(\omega; Q)$; for the set $E-E_0$ is countable as subset of H.

Let $\eta(u)$ be a fixed right-hand direction curve of ω in what follows. This is plainly feasible since ω is light. In view of the relation $\Omega_*(\omega; N)=0$ proved above it follows from part (i) of the theorem of the foregoing section that, at each point u_1 of Q, the curve ω possesses a tangent direction equal to $\eta(u_1)$. On the other hand φ is continuous at all points of E_0 and so, writing $t_1=p(u_1)$, we readily see that $\gamma(t_1)$, which is by hypothesis a derived direction of φ at t_1, must also be a derived direction of ω at u_1. Accordingly $\gamma(t_1)=\eta(u_1)$. We have thus shown that $\gamma(t)=\eta(q(t))$ for each $t\in E_0$. As a direct consequence we find that $\Xi(\gamma; E)=\Xi(\gamma; E_0)=\Xi(\eta; Q)$.

Now $\Lambda_*(\eta; X)=\Omega_*(\omega; X)$ for every set $X\subset \boldsymbol{R}$ (and in particular for $X=Q$) by the lemma of [5]§3. But we know already that ω is straightenable and that $\Omega_*(\omega; N)$ vanishes for countable sets $N\subset Q$. It ensues that η is rectifiable and moreover continuous at all points of Q, so that we deduce, using the theorem of [3]§4 and the lemma of [5]§2, that

$$\Xi(\eta; Q)=L_*(\eta; Q)=\Lambda_*(\eta; Q)=\Omega_*(\omega; Q).$$

Similarly we get $\Xi(\gamma; E)=L_*(\gamma; E)$. Combining the last two relations with what has already been proved we are finally lead to $\Omega_*(\varphi; E)=L_*(\gamma; E)$, which completes the proof.

5. Expression of reduced measure-bend as reduced measure-length. We are now prove the following result.

THEOREM. *Given a light curve φ and a set E, let γ be any direction curve (on $\boldsymbol{R}$) of φ and suppose that $\Omega_*(\varphi; \{t\})$ is finite for every point t of E. (The latter condition is certainly fulfilled when φ is locally straightenable.) Then we have $\Xi(\gamma; E)=\Upsilon(\varphi; E)$.*

PROOF. Assuming E nonvoid as we may, we define for each $n=1, 2, \cdots$ an open set U_n of real numbers in exactly the same way as at the beginning of the foregoing proof. Then, since Ξ and Υ are both outer Carathéodory measures, $\Xi(\gamma; EU_n)$ and $\Upsilon(\varphi; EU_n)$ tend as $n\to+\infty$ to the respective limits $\Xi(\gamma; E)$ and $\Upsilon(\varphi; E)$. Consequently it is enough to prove $\Xi(\gamma; EU_n)=\Upsilon(\varphi; EU_n)$ for each n.

The inequality $\Omega_*(\varphi; U_n)\leqq\Omega(\varphi; I_1)+\cdots+\Omega(\varphi; I_n)$, where the intervals $I_1, I_2, \cdots$ mean the same things as in §4, implies that $\Omega_*(\varphi; U_n)<+\infty$. If, therefore, we write K_n for the set of the points t of EU_n such that $\Omega_*(\varphi; \{t\})=0$, it is obvious that EU_n-K_n is countable. It follows at once that $\Xi(\gamma; EU_n)=\Xi(\gamma; K_n)$ and $\Upsilon(\varphi; EU_n)=\Upsilon(\varphi; K_n)$.

Now $\Upsilon(\varphi; K_n)=\Omega_*(\varphi; K_n)$ by the theorem of [6]§2, while $\Xi(\gamma; K_n) =L_*(\gamma; K_n)$ by the remark of §1 and the lemma of [5]§3. Thus the proof reduces to showing $L_*(\gamma; K_n)=\Omega_*(\varphi; K_n)$. But this certainly holds in virtue of the preceding theorem.

6. Supplement. In connection with part (ii) of the theorem of §3 we can state the following result, the proof of which will be given in our forthcoming note.

THEOREM. *A light curve φ is spherically representable on both sides* (see [1]§77) *provided that it is locally straightenable.*

References

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