34. On the Behaviour of Analytic Functions on the Ideal Boundary. I

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The present paper is an application of the previous papers "Potentials on Riemann surfaces" and "Singular points of Riemann surfaces"[1] which we abbreviate by P and S respectively. Notations and terminologies are to be referred to them.

Let R be a Riemann surface with positive boundary. Let $R_n(n=0,1,2,\cdots)$ be an exhaustion with compact relative boundary ∂R_n. Let $N(z,p)$ be an N-Green's function. We suppose that N-Martin's topology is defined on $R-R_0+B^N$, where B^N is the ideal boundary of R obtained by the completion of $R-R_0$ with respect to N-Martin's topology. We denote by B_1^N the set of N-minimal boundary point. Then $B_0^N=B^N-B_1^N$ is an F_σ set of *capacity zero*. Let G be a domain[2] in $R-R_0$ and let ${}_{CG}N(z,p): p\in B_1^N$ be the least positive superharmonic function in $R-R_0$ with ${}_{CG}N(z,p)=N(z,p)$ on CG $[{}_{CG}N(z,p) =\lim\limits_{M=\infty} U^M(z)$, where $U^M(z)$ is a harmonic function in G such that $U^M(z)=\min(M, N(z,p))$ on CG and $U^M(z)$ has M.D.I. (Minimal Dirichlet Integral) over $G]$. If $N(z,p)>{}_{CG}N(z,p)$, we say that G contains p *N-approximately* and denote it by $G\overset{N}{\ni}p$.

Let $G(z,p)$ be a Green's function of R. Put $K(z,p)=\dfrac{G(z,p)}{G(p_0,p)}$, where p_0 is a fixed point. We suppose that K-Martin's topology is defined in $R+B^K$ by use of $K(z,p)$, where B^K is the ideal boundary. Let B_1^K be the set of K-minimal boundary points of R. Then B_0^K $B^K-B_1^K$ is an F_σ set of *harmonic measure zero*. Let G be a domain in R and let $K_{CG}(z,p)$ be the least positive superharmonic function in R with $K_{CG}(z,p)=K(z,p): p\in B_1^K$ on CG. If $K(z,p)>K_{CG}(z,p)$, we say that G contains p *K-approximately* and denote it by $G\overset{K}{\ni}p$. Then we have the following

Lemma 1. a). 1). *If* $G_i\overset{N}{\ni}p: i=1,2,\cdots,l$, $\bigcap\limits^{l} G_i\overset{N}{\ni}p$. 2). *If* $G\overset{N}{\ni}p$, $(\operatorname{int} CG)\overset{N}{\not\ni}p$. 3). $E\left[z\in R+B^N,\ \operatorname{dist}(z,p)<\dfrac{1}{n}\right]=v_n(p)\overset{N}{\ni}p$.

1) Z. Kuramochi: Potentials on Riemann surfaces; Singular points: Journ. Sci. Hokkaido Univ. **14** (1962).

2) We suppose that ∂G consists of at most enumerably infinite number of analytic curves clustering nowhere in R.

b). 1). *If* $G_i \overset{K}{\ni} p: i=1,2,\cdots,l$, $\bigcap^{l} G_i \overset{K}{\ni} p$. 2). *If* $G \overset{K}{\ni} p$, (int $CG \overset{K}{\not\ni} p$.
3). $E\left[z \in R+B^K,\ \text{dist}\,(z,p)<\frac{1}{n}\right]=v_n(p) \overset{K}{\ni} p$.

Proof of a). *Case 1*). p is singular: Cap$\,(p)>0$. In this case the proof of a) is given in Theorem 8 of S.

Case 2). p is not singular: Cap$\,(p)=0$ and $p\in B_1^N$. At first we show ${}_p({}_{CG}N(z,p)=0$. Since p is one point $\in B_1^N$, ${}_p({}_{CG}N(z,p))=a_0N(z,p)$: $a_0\geqq 0$. Since Cap$\,(p)=0$, $N(z,p)-{}_p({}_{CG}N(z,p))=U_0(z)$ is $\overline{\text{super}}$harmonic by Theorem 6 of P and by mass of $U_0(z)\leqq 1$ we have $D(\min(M,U(z))\leqq 2\pi M$. Put

$${}_{CG}N(z,p)=a_0N(z,p)+U_0(z),\quad {}_pU_0(z)=a_1N(z,p),$$

$$U_n(z)=a_{n+1}N(z,p)+U_{n+1}(z), \text{ where } {}_p(U_n(z))=a_{n+1}N(z,p): n=1,2,\cdots$$

Then $D(\min(M,U_n(z))\leqq 2\pi M$, $U_n(z)$ is $\overline{\text{super}}$harmonic, $U_n(z)\downarrow U_\infty(z)$ and $U_\infty(z)$ is also $\overline{\text{super}}$harmonic by Theorem 4 of P. By $N(z,p)\geqq {}_{CG}N(z,p)=\sum^{\infty} a_iN(z,p)+U_\infty(z)$ we have $\sum^{\infty} a_i\leqq 1$ and $\lim_n a_n=0$ and

$${}_p(U_\infty(z))=\lim a_nN(z,p)=0.$$

Suppose $G \overset{N}{\ni} p$. Then $N(z,p)>{}_{CG}N(z,p)$ and $\sum^{\infty} a_i<1$, whence $U_\infty(z)>0$. Now $U_\infty(z)=(1-\sum^{\infty} a_i)N(z,p)$ on CG and $U_\infty(z)$ is $\overline{\text{super}}$harmonic. On the other hand, ${}_{CG}N(z,p)$ is the least positive $\overline{\text{super}}$harmonic function with ${}_{CG}N(z,p)=N(z,p)$ on CG. Hence $U_\infty(z)\geqq(1-\sum^{\infty} a_i)_{CG}N(z,p)$ and ${}_p({}_{CG}N(z,p))=0$. Suppose $G_i \overset{N}{\ni} p$. Then ${}_p({}_{CG_i}N(z,p))=0$ and ${}_p({}_{\sum_i CG_i}N(z,p))\leqq\sum_i {}_p({}_{CG_i}N(z,p))=0$. Hence $N(z,p)={}_pN(z,p)>{}_p({}_{\sum_i CG_i}N(z,p))$, whence $N(z,p)>{}_{\Sigma CG_i}N(z,p)$. Thus $\bigcap^{l} G_i \overset{N}{\ni} p$. If $G \overset{N}{\ni} p$, ${}_p({}_{CG\cap p}N(z,p))\leqq {}_p({}_{CG}N(z,p))=0$. Similarly by int $CG \overset{N}{\ni} p$ ${}_p({}_{G\cap p}N(z,p))=0$. Now $N(z,p)={}_pN(z,p)={}_{p\cap CG}N(z,p)+{}_{p\cap G}N(z,p)=0$. This is a contradiction. Hence we have a). 2). The proof of a). 3) is given in Theorem 19 of S.

Proof of b). Let $U(z)$ be a positive superharmonic function and let F be a closed set of B^K. Then $U(z)-U_F(z)$ is superharmonic i.e. $U(z)-U_p(z)$ is superharmonic (F is not necessarily a closed set of harmonic measure zero). Hence we have b). 1 and b). 2 similarly as a). The proof of b). 3) is given in Theorem 3 of S.

Let $w=f(z): z\in R$ be an analytic function whose values fall on the w-Riemann sphere. If the spherical area $A(f(z))$ of the image of R by $w=f(z)$ is finite, we call $f(z)$ a function *of D-type*. Map the universal covering surface R^∞ of R onto $|\zeta|<1$ conformally by $z=z(\zeta)$. If the function $w=f(z(\zeta))=f(\zeta)$ has angular limits a.e. on $|\zeta|=1$, we call $f(z)$ a function *of F-type*. It is well known, if $w=f(z)$ is of bounded type (the characteristic function of $T(z)$ of $f(z)$ is bounded), $T(\zeta)\leqq T(z)$[3] and $w=f(\zeta)$ is of F-type, where $T(\zeta)$ is the characteristic of $w=f(\zeta)$. Put $M^N(p)=\bigcap\overline{f(G_i)}:G_i\overset{N}{\ni}p$ and $M^K(p)=\bigcap\overline{f(G_i)}:G_i\overset{K}{\ni}p$, where the intersection is taken over all domains containing $N(K)$-approximately. We see that by Lemma 1 that $M^N(p)(M^K(p))$ is closed and consists of only one point or of a continuum. Then we have the following

Theorem a). *Denote by $S^N(S^K)$ the set of points such that $M^N(p)(M^K(p))$ is a continuum. Then if $w=f(z)$ is of D-type, $S^N+S^N_0$ does not contain any closed set of positive capacity.*

b). *If $w=f(z)$ is of F-type, $S^K+B^K_0$ does not contain any closed set of positive harmonic measure.*

Lemma 2. a). *Let G and $G':G\supset G'$ be domains such that $D(\omega^*(z))<\infty$, where $\omega^*(z)$ is a harmonic function in $G-G'$ such that $\omega^*(z)=0$ on ∂G, $\omega^*(z)=1$ on $\partial G'$ and $\omega^*(z)$ has* M.D.I. *Let F be a closed subset of B^N. Then we can define* C.P. *of $F\cap G'$ relative to $G:\omega(F\cap G',z,G)=\lim\limits_n\lim\limits_i \omega_{n,n+i}(z)$, where $\omega_{n,n+i}(z)$ is a harmonic function in $(G\cap R_{n+i})-(F_n\cap G'):F_n=E\left[z\in R+B^N,\ \mathrm{dist}\,(F,z)\leqq\frac{1}{n}\right]$ such that $\omega_{n,n+i}(z)=0$ on $\partial G\cap R_{n+i}$, $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $(\partial R_{n+i}\cap G)-(F_n\cap G')$, $\omega_{n,n+i}(z)=1$ on $F_n\cap G'$. If $\omega(F\cap G',z,G)>0$, G contains at least one point $p\in F\cap B^N_1$ N-approximately.*

b). *Let G be a domain and let F be a closed set of B^K. If $w(z,F,G)=\lim\limits_n\lim\limits_i w_{n,n+i}(z)>0$, G contains at least one point $p\in B^K_1\cap F$ K-approximately, where $w_{n,n+i}(z)$ is a harmonic function in $(G\cap R_{n+i})-F_n$ such that $w_{n,n+i}(z)=0$ on $(\partial G\cap R_{n+i})+\partial R_{n+i}-F_n$ and $w_{n,n+i}(z)=1$ on $F_n=E\left[z\in R+B^K,\ \mathrm{dist}\,(z,F)\leqq\frac{1}{n}\right]$.*

Proof of a). For the simplicity put $\omega(z)=\omega(F\cap G',z,G)$. Assume $\omega(z)>0$. Put $D=D(\omega(z))$ $(<\infty)$. Then $\omega(z)$ has M.D.I. over $\Omega=E[z,\delta_1<\omega(z)<\delta_2]:0<\delta_1<\delta_2<1$ and there exists a *regular niveau* curve C_δ for almost all $\delta:C_\delta=E[z,\omega(z)=\delta]$. Put $U(z)={}_{CG}\omega(z,F)$ ($\omega(F,z)$ is C.P. of F). Then $U(z)$ has M.D.I. over G, whence $U(z)$ has M.D.I. over Ω. Hence $U(z)=\lim\limits_n U_n(z)$, where $U_n(z)$ is a harmonic function in $R_n\cap\Omega$ such that $U_n(z)=U(z)$ on C_{δ_i} $(i=1,2)$ $\frac{\partial}{\partial n}U_n(z)=0$ on $\Omega\cap\partial R_n$, where C_{δ_i} is regular. Then by the Green's formula

$$\int_{C_{\delta_1}}U_n(z)\frac{\partial}{\partial n}\omega_n(z)\,ds=\int_{C_{\delta_2}}U_n(z)\frac{\partial}{\partial n}\omega_n(z)\,ds\,,$$

where $\omega_n(z)$ is a harmonic function in $\Omega\cap R_n$ such that $\omega_n(z)=\delta_i$ on

3) Z. Kuramochi: Dirichlet problem on Riemann surfaces. 1, Proc. Japan Acad., **30**, 731-735 (1954).

C_{δ_i} and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\partial R_n \cap \Omega$ and $\lim_n \omega_n(z)=\omega(z)$. Let $n\to\infty$. Then by Theorem 3 of P we have $\int_{C_{\delta_1}} U(z)\frac{\partial}{\partial n}\omega(z)\,ds=\int_{C_{\delta_2}} U(z)\frac{\partial}{\partial n}\omega(z)\,ds$, $D=\int_{C_{\delta_i}}\frac{\partial}{\partial n}\omega(z)\,ds: i=1,2$. Now $U(z)<1$ in $R-R_0$, whence there exists a positive number ε_0 such that

$\int_{C_{\delta_1}} U(z)\frac{\partial}{\partial n}\omega(z)\,ds<D(1-\varepsilon_0)$. Let $\delta_2\to 1$. Then

$$\int_{C_{\delta_2}} U(z)\frac{\partial}{\partial n}\omega(z)\,ds<D(1-\varepsilon_0)<\int_{C_{\delta_2}} \omega(z)\frac{\partial}{\partial n}\omega(z)\,ds=D\delta_2 \text{ for } \delta_2>1-\varepsilon_0.$$

Whence $\quad {}_{CG}\omega(F,z)<\omega(F\cap G',z,G)\leqq\omega(F,z)$ in G.

Now by Theorem 13 of P $\omega(F,z)$ is represented by a positive mass on $F\cap B_1^N$: $\omega(F,z)=\int N(z,p)\,d\mu(p)$. And by Theorem 4 of P ${}_{(CG)_n}\omega(F,z)\uparrow\omega(F,z)$, where $(CG)_n=CG\cap R_n$. Since $N(z,p)$ is *uniformly continuous with respect to* p in *every compact set* not containing p, $N(z,p_i)\to N(z,p)$ on $(CG)_n$ as $p_i\to p$. Now $N(z,p_i)$ and $N(z,p)$ are harmonic in $R-R_0-(CG)_n$, whence by the $\overset{*}{m}$*aximum principle*

$$\max_{z\in R-R_0-(CG)_n}|N(z,p_i)-N(z,p)|\leqq\max_{z\in\partial(CG)_n}|N(z,p_i)-N(z,p)|.$$

Hence we can find a sequence of linear forms $V_m(z)=\sum^m c_iN(z,p_i)$: $c_i>0: m=1,2,\cdots$ such that $V_m(z)\to\omega(F,z)$ uniformly on $(CG)_n$ and ${}_{(CG)_n}V_m(z)=\sum^m c_i\,{}_{(CG)_n}N(z,p)\to\int{}_{(CG)_n}N(z,p)\,d\mu(p)$ uniformly on $(CG)_n$ as $m\to\infty$, whence by the $\overset{*}{m}$aximum principle ${}_{(CG)_n}V_m(z)\to\int{}_{(CG)_n}N(z,p)\,d\mu(p)$ not only on $(CG)_n$ but also on $(CG)_n+(R-R_0-(CG)_n)=R-R_0$, because ${}_{(CG)_n}(\sum^m c_iN(z,p_i))=\sum^m c_i\;{}_{(CG)_n}N(z,p_i)$ is clear. Let $m\to\infty$. Then

$${}_{(CG)_n}\omega(F,z)={}_{(CG)_n}\Big(\int N(z,p)\,d\mu(p)\Big)=\int{}_{(CG)_n}N(z,p)\,d\mu(p).$$

Let $n\to\infty$. Then by ${}_{(CG)_n}N(z,p)\uparrow{}_{CG}N(z,p)$ we have ${}_{CG}\omega(F\,z,)$ $=\int{}_{CG}N(z,p)d\mu(p)$ and

$$\omega(F,z)=\int N(z,p)\,d\mu(p)>\int{}_{CG}N(z,p)\,d\mu(p)={}_{CG}\omega(F,z).$$

Hence there exists at least a point $p\in F\cap B_1^N$ such that $N(z,p)>{}_{CG}N(z,p)$, i.e. $G\overset{N}{\ni}p$.

Proof of b). Assume $w(F,z,G)>0$. Put $w(\partial G,z,G)=1-w(z,G\cap B^K,G)$ $(\leqq 1-w(F,z,G))$. Put $G_{1-\varepsilon}=E[z\in G, w(F,z,G)<1-\varepsilon]$ and $G'_\varepsilon=E[z\in G, w(\partial G,z,G)>\varepsilon]: \frac{1}{3}>\varepsilon>0$. Then by P.H.3 of P $w(F\cap G_{1-\varepsilon},z,G,=0=w(G'_\varepsilon\cap B^K,z,G)\geqq w(F\cap G'_\varepsilon,z,G)$. Hence by $w(F\cap(G_{1-\varepsilon}+G'_\varepsilon),z,G)\leqq w(F\cap G_{1-\varepsilon},z,G)+w(F\cap G'_\varepsilon,z,G)=0$ and by $w(F\cap(C(G_{1-\varepsilon}$

$+G'_\varepsilon)), z, G)+w(F\cap(G_{1-\varepsilon}, z, G'_\varepsilon), G)\geqq w(F, z, G)>0$ we have $w(F\cap CG_{1-\varepsilon}\cap CG'_\varepsilon, z, G)>0$ and $CG_{1-\varepsilon}\cap CG'_\varepsilon$ is non void. Whence $w(F, z)-w_{CG}(F, z)\geqq w(F, z, G)-w(\partial G, z, G)\geqq 1-2\varepsilon>0$ in $CG_{1-\varepsilon}\cap CG'_\varepsilon$. Hence $w(F, z)\geqq w_{CG}(F, z)$. Thus we have (b) as above.

We denote by $\delta M(p)$ the spherical diameter of $M(p)$. Let C be a circle in the w-sphere. Then $f^{-1}(C)$ consists of enumerably infinite number of domains. Then by the definition of $\delta M(p)$ we have easily the following

Lemma 3. *If* $\text{dia}\,(C)<\delta_0$, *any domain of* $f^{-1}(C)$ *does not contain* N*(or* K*) approximately a point such that* $\delta M^N(p)(\delta M^K(p))>\delta_0$.

We have by Lemma 1 the following

Lemma 4. *If* $M(p)=q$ *(one point), there exists an asymptotic path* L *tending to* p *on which* $f(z)\to q$ *as* $z\to p$.

Lemma 5. *Let* G *be a domain in* R. *Then* $E[p\in B^L_1, p\overset{L}{\notin} G]$ *is a* G_δ *set in* B^L_1. *Let* $C_{n,i}$ $(i=1,2,\cdots)$ *be a system of spherical circles with radius* $\frac{1}{n}$ *such that any circle with radius* $\frac{1}{3n}$ *is contained in a certain* $C_{n,i}$. *Put* $T^L_{n,i}=E[p\in B^L_1, p\overset{L}{\notin}$ *any component of* $f^{-1}(C_{n,i})]$ *and* $S^L_n=E\left[p\in B^L_1, \delta M(p)\geqq\frac{1}{n}\right]$. *Then* $\bigcup_n S^L_n=\bigcup_n(\bigcap_i T^L_{n,i})$ *is a* $G_{\delta\sigma}$ *set in* B^L_1, *where* $L=N$ *or* K.

Proof. By the compactness of $(CG)_n$ ${}_{(CG)_n}N(z, p_i)\to{}_{(CG)_n}N(z, p)$ as $p_i\to p$ and by ${}_{(CG)_n}N(z, p)\uparrow{}_{CG}N(z, p)$ as $N\to\infty$, $N(z, p)-{}_{CG}N(z, p)$ is upper semicontinuous and $E[p\in B^L_1, N(z, p)-{}_{CG}N(z, p)=0]$ is a G_δ set in B. Now $f^{-1}(C_{n,i})$ consists of at most enumerably infinite number of domains. Hence $T^N_{n,i}$ is also a G_δ set. Clearly by $\text{dia}\,(C_{n,i})=\frac{1}{n}$ $S_n\subset\bigcap_i T_{3n,i}$. Next assume $p\notin S^N_{3n}$. Then $\delta M(p)<\frac{1}{3n}$ and there exists a domain G such that $\text{dia}\,(f(G))=\frac{1}{3n}$, $G_{n,i}\overset{N}{\ni}p$ and a circle $C_{n,i}\supset f(G)$. Hence $p\overset{N}{\notin}\bigcap_i T_{n,i}$ and $S^N_{3n}\supset\bigcap_i T^N_{n,i}$ and $\bigcup S^N_n=\bigcup_n(\bigcap_i T_{n,i})$. Above facts for $K(z, p)$ are proved similarly.

Proof of the theorem. Proof of a). Assume that there exists a number n_0 such that $\bigcap_i T^N_{n_0,i}+B^N_0$ has a closed set F of positive capacity. Let C_i be a circle with radius $\frac{1}{5n_0}$ such that $\sum_i \text{int}\, C_i$ covers the w-sphere. Then by $\sum_{i,j}G_{i,j}\supset R-R_0$ ($G_{i,j}$ is a component of $f^{-1}(C_i)$) and by $\sum_{i,j}\omega(G\cap G_{i,j}\cap F, z)\geqq\omega(F, z)$, there exists at least one domain G of $G_{i,j}$ such that $\omega(F\cap G, z)>0$, where $\omega(F\cap G, z)$ is C.P. of $F\cap G$. Let $\widetilde{C}_i$ be a spherical circle with the same centre as that of C_i and with radius $\frac{2}{5n_0}$. Now by a rotation of the w-sphere we can suppose without loss of generality that the closure of $\tilde{C}_i$ does not contain the north pole of the sphere. Let $\tilde{G}$ be a component of $f^{-1}(\tilde{C}_i)$ containing G. Let $U(z)$ be a continuous function in the w-sphere snch that $U(z)$ is harmonic in $\tilde{C}_i - C_i$, $U(z)=0$ on the complementary set of $\tilde{C}_i$ and $U(w)=1$ on the closure of C_i. Then max $\left(\left|\frac{\partial U(w)}{\partial u}\right|, \left|\frac{\partial U(w)}{\partial v}\right|\right)=M<\infty : w=u+iv$. Consider the function $U(z) = U(f^{-1}(w)) : z \in R-R_0$. Then $D(U(z))<M^2AK$, where A is the spherical area of the image of R and $K=$maximum of $\frac{\text{area element}}{\text{spherical area element}}$ in $\bar{\tilde{C}}_i$ (clearly $K<\infty$). Let $\omega_n(z)$ be a continuous function in $R-R_0$ such that $\omega_n(z)$ is harmonic in $R-R_0-(F_n \cap G)\left(F_n=E\left[z \in R+B^N, \text{dist}\,(z, F) \leqq \frac{1}{n}\right]\right)$, $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)=1$ on $(F_n \cap G)$ and has M.D.I. Then there exists a number n' such that $D(\omega_n(z))<L<\infty$ for $n \geqq n'$ and $0<\omega(F \cap G, z)=\lim_n \omega_n(z)$. Put $\tilde{\omega}_n(z)=\min\,(U(z), \omega_n(z))$. Then $\tilde{\omega}_n(z)=0$ on $\partial R_0+\partial\tilde{G}$, $\tilde{\omega}_n(z)=1$ on $(F_n \cap G)$ and $D(\tilde{\omega}_n(z)) \leqq D(\omega_n(z))+D(U(z))<L'<\infty$ for $n \geqq n'$. Let $\tilde{\tilde{\omega}}_n(z)$ be a harmonic function in $\tilde{G}-(F_n \cap G)$ such that $\tilde{\tilde{\omega}}_n(z)=0$ on $\partial\tilde{G}$, $\tilde{\tilde{\omega}}(z)=1$ on $F_n \cap G$ and has M.D.I. Then by the Dirichlet principle

$$0<D(\omega(F \cap G, z)) \leqq D(\tilde{\tilde{\omega}}_n(z)) \leqq D(\tilde{\omega}_n(z))<L' \quad \text{for } n \geqq n'.$$

Hence $\lim_n \tilde{\tilde{\omega}}_n(z)=\omega(F \cap G, z, \tilde{G})>0$. Hence by Lemma 2 $\tilde{G}$ contains at least one point $p \in F \cap B_1^N N$-approximately, on the hand, by Lemma 3 G does not contain any point of F by dia$\,(\tilde{C})=\frac{4}{5n_0}<\delta M(p) \geqq \frac{1}{n_0}$. This is a contradiction. Hence we have a) by Lemma 5.

Proof of b). Assume $\bigcap_i T^K_{n_0,i}+B^K_0$ has a closed set F of positive harmonic measure. Then similarly as above, we can find domains G and $\tilde{G}$ such that $w(F \cap G, z)>0$ and $\text{dist}\,(f(G), f(\partial\tilde{G})) \geqq \frac{1}{5n_0}>0$. Since $f(\zeta)$ is of F-type, we have mes $E>0$ by $w(F \cap G, z)>0$, where E is the set on $|\zeta|=1$ on which $f(\zeta)$ has angular limits contained in $\overline{f(G)}$ and $w(F \cap G, z)=1$ a.e. on E. Now $f(\partial\tilde{G})$ does not tend to E by $\text{dist}\,(f(G), f(\tilde{G}))>0$ and we see $\omega(F \cap G, z, \tilde{G})>0$. Thus we have b) similarly as a).