77. *On Unified Representation of State Vector in Quantum Field Theory*

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1. **Introduction.** In quantum field theory we must consider the Hilbert space having non-countable bases which corresponds to a sequence of non-negative integers $(n_1, n_2, \cdots)$.

Since we can construct one-to-one mapping from the set of the sequences $(n_1, n_2, \cdots)$ onto the points in $[0, 1]$ interval [8], we can identify these bases to $[0, 1]$ interval.

Let γ denote a point in $[0, 1]$ interval and let ψ_γ be the element of the Hilbert space which corresponds to γ. The element of this Hilbert space is usually represented by the formulae $\int C_\gamma \psi_\gamma d\mu(\gamma)$ and $\sum_{i=1}^{\infty} C_i \psi_{\gamma_i}$, in [3], [4] and [6], where C_i, C_γ are constants, and $d\mu(\gamma)$ is a measure on $[0, 1]$.

By single $d\mu(\gamma)$, however, we cannot represent every element of this Hilbert space. That is to say, by a continuous measure $d\mu(\gamma)$, we cannot represent the element of the second form. On the other hand by the second form, we cannot represent the element of the first form.

In this paper we take a Lebesgue measure $dm(\gamma)$ and represent each element of the Hilbert space by the unified single expression $\int (C_\gamma + C'_\gamma \sqrt{\delta_\gamma})\, dm(\gamma)$ using generalized distributions [7].

Our method of representation uses a L^2-space's closure. But our topology is weaker than L^2-topology.

2. **New topology defined in $L^2[0, 1]$.**

Lemma 1. *There is a one-to-one correspondence between the sequence of non-negative integers $(n_1, n_2, \cdots)$ and the point of interval $[0, 1]$.* [8]

Let's consider the corresponding interval $[0, 1]$. Let $L^2[0, 1]$ denote the space of functions which are defined in the interval $[0, 1]$ and belong to L^2.

Let $\rho_{n,x_0}(x)$ denote the function

$$\rho_{n,x_0}(x) = \begin{cases} 0 & \text{for } |x-x_0| \geqq \delta/n \\ kn \exp\{-(\delta/n)^2/((\delta/n)^2 - |x-x_0|^2)\}/\delta & \text{for } |x-x_0| < \delta/n, \end{cases}$$

where δ is a positive constant and k is a constant which satisfies the following equality: $k\int_{|x|<1} \exp\{-1/(1-x^2)\}\, dx = 1$.

In the space $L^2[0,1]$, we introduce the new topology by the following neighbourhoods:

$$U_\varepsilon(\psi)=\left[\begin{array}{l} \varphi;\ \sup_x \left|\int_0^x\int_0^s \{|\mathfrak{R}_+\varphi(t)|^2-|\mathfrak{R}_+\psi(t)|^2\}\,dtds\right|<\varepsilon, \\ \sup_x \left|\int_1^x\int_1^s \{|\mathfrak{R}_+\varphi(t)|^2-|\mathfrak{R}_+\psi(t)|^2\}\,dtds\right|<\varepsilon, \\ \sup_x \left|\int_0^x\int_0^s \{|\mathfrak{R}_-\varphi(t)|^2-|\mathfrak{R}_-\psi(t)|^2\}\,dtds\right|<\varepsilon, \\ \sup_x \left|\int_1^x\int_1^s \{|\mathfrak{R}_-\varphi(t)|^2-|\mathfrak{R}_-\psi(t)|^2\}\,dtds\right|<\varepsilon, \\ \sup_x \left|\int_0^x\int_0^s \{|\mathfrak{I}_+\varphi(t)|^2-|\mathfrak{I}_+\psi(t)|^2\}\,dtds\right|<\varepsilon, \\ \sup_x \left|\int_1^x\int_1^s \{|\mathfrak{I}_+\varphi(t)|^2-|\mathfrak{I}_+\psi(t)|^2\}\,dtds\right|<\varepsilon, \\ \sup_x \left|\int_0^x\int_0^s \{|\mathfrak{I}_-\varphi(t)|^2-|\mathfrak{I}_-\psi(t)|^2\}\,dtds\right|<\varepsilon, \\ \sup_x \left|\int_1^x\int_1^s \{|\mathfrak{I}_-\varphi(t)|^2-|\mathfrak{I}_-\psi(t)|^2\}\,dtds\right|<\varepsilon \end{array}\right]$$

where
$$\mathfrak{R}_+\varphi(t)=\{|\varphi(t)+\overline{\varphi(t)}|/2+(\varphi(t)+\overline{\varphi(t)})/2\}/2,$$
$$\mathfrak{R}_-\varphi(t)=\{|\varphi(t)+\overline{\varphi(t)}|/2-(\varphi(t)+\overline{\varphi(t)})/2\}/2,$$
$$\mathfrak{I}_+\varphi(t)=\{|\varphi(t)-\overline{\varphi(t)}|/2+(\varphi(t)-\overline{\varphi(t)})/(2i)\}/2,$$
$$\mathfrak{I}_-\varphi(t)=\{(\varphi(t)-\overline{\varphi(t)})/(2i)-|\varphi(t)-\overline{\varphi(t)}|/2\}/2.$$

Lemma 2. *The space $L^2[0,1]$ is a Hausdorff space.*

Proof. It is evident that the axioms (A) $\psi\in U_\varepsilon(\psi)$ and (B) $U_{\min(\varepsilon_1,\varepsilon_2)}(\psi)\subseteq U_{\varepsilon_1}(\psi)\cap U_{\varepsilon_2}(\psi)$ are satisfied.

We see also that the following inequality is satisfied; for $\varphi_1\in U_{\varepsilon_1}(\psi)$, $\varphi_2\in U_{\varepsilon_2}(\varphi_1)$,

$$\begin{aligned} &\left|\int_0^x\int_0^s \{|\mathfrak{R}_+\varphi_2(t)|^2-|\mathfrak{R}_+\psi(t)|^2\}\,dtds\right| \\ \leqq &\left|\int_0^x\int_0^s \{|\mathfrak{R}_+\varphi_2(t)|^2-|\mathfrak{R}_+\varphi_1(t)|^2\}\,dtds\right| \\ +&\left|\int_0^x\int_0^s \{|\mathfrak{R}_+\varphi_1(t)|^2-|\mathfrak{R}_+\psi(t)|^2\}\,dtds\right| \\ <&\ \varepsilon_1+\varepsilon_2. \end{aligned}$$

By the same way, we can prove other similar inequalities for $\mathfrak{R}_-$, $\mathfrak{I}_+$, $\mathfrak{I}_-$ and $\int_1^x\int_1^s\cdots dtds$. Now, if $\varphi_1\in U_\varepsilon(\psi)$, then $\varphi_1\in U_{\varepsilon_1}(\psi)$ for $0<\varepsilon_1<\varepsilon$. So, if we take $0<\varepsilon_2<\varepsilon-\varepsilon_1$ then $U_{\varepsilon_2}(\varphi_1)\subset U_\varepsilon(\psi)$. Hence we see that the axiom (C) is satisfied.

If $\varphi_1(t)\neq\psi(t)$ in the sense of $L^2[0,1]$, then at least one of the following inequalities is satisfied;

$$\left|\int_0^x\int_0^s \{|\Re_\pm\varphi_1(t)|^2-|\Re_\pm\psi(t)|^2\}\,dtds\right|\geqq\varepsilon,$$

$$\left|\int_1^x\int_1^s \{|\Re_\pm\varphi_1(t)|^2-|\Re_\pm\psi(t)|^2\}\,dtds\right|\geqq\varepsilon,$$

$$\left|\int_0^x\int_0^s \{|\Im_\pm\varphi_1(t)|^2-|\Im_\pm\psi(t)|^2\}\,dtds\right|\geqq\varepsilon,$$

$$\left|\int_1^x\int_1^s \{|\Im_\pm\varphi_1(t)|^2-|\Im_\pm\psi(t)|^2\}\,dtds\right|\geqq\varepsilon.$$

(In this representation, double suffices $\pm$ are taken by the same order.) So, the axiom (D) is satisfied. Therefore $\boldsymbol{L}^2[0,1]$ is a Hausdorff space.

Further we can see in III that the topology of this space is uniform [9]. Let $\overline{\boldsymbol{L}^2[0,1]}$ denote the closure of the space $\boldsymbol{L}^2[0,1]$ by this topology.

Remark. We can also express this topology by the following way.

Let's decompose f_n into four parts: $f_n=f_n^{\Re+}-f_n^{\Re-}+if_n^{\Im+}-if_n^{\Im-}$. ($f_n^{\Re+}, f_n^{\Re-}, f_n^{\Im+}, f_n^{\Im-}$ are positive distributions.) We call $\{f_n\}$ converges to $\sqrt{T^{\Re+}}-\sqrt{T^{\Re-}}+i\sqrt{T^{\Im+}}-i\sqrt{T^{\Im-}}$ in the new topology if and only if $\lim_{n\to\infty}(f_n^{\Re+})^2=T^{\Re+}$, $\lim_{n\to\infty}(f_n^{\Re-})^2=T^{\Re-}$, $\lim_{n\to\infty}(f_n^{\Im+})^2=T^{\Im+}$, and $\lim_{n\to\infty}(f_n^{\Im-})^2=T^{\Im-}$ in D$'$ topology.

3. **Classification of the Cauchy sequences.** The Cauchy sequences $\{\varphi_n\}$ in $\boldsymbol{L}^2[0,1]$ satisfy the following condition; for an arbitrary $\varepsilon>0$, there exists a sufficiently large number N such that $\varphi_n\in U_\varepsilon(\varphi_m)$ for all $m,n>N$.

We classify these sequences by the following equivalence relations $\simeq$: the Cauchy sequence $\{\varphi_n\}$ is equivalent to $\{\psi_n\}$ if and only if there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n>0$, $\lim_{n\to\infty}\varepsilon_n=0$ and $\varphi_m\in U_{\varepsilon_n}(\psi_{m'})$, $\psi_{m'}\in U_{\varepsilon_n}(\varphi_m)$ for $m, m'>N$.

We denote this relation by $\{\varphi_n\}\simeq\{\psi_n\}$. If we define the equivalent class of the Cauchy sequences by this topology, then we can identify the set of these classes to the complete space $\overline{\boldsymbol{L}^2[0,1]}$.

Lemma 3. *If* $\varphi_n\in U_\varepsilon(\psi)$, $\varphi_m\in U_\varepsilon(\psi)$; *then*

(1) $\varphi_n+\varphi_m\in U_{4\varepsilon}(2\psi)$,

(2) $\alpha\varphi_n\in U_{|\alpha|^2\varepsilon}(\alpha\psi)$,

(3) $\alpha\varphi_n+\beta\varphi_m\in U_\varepsilon(\psi)$ *for* $\alpha+\beta=1$, $0\leq\alpha\leq1$, $0\leq\beta\leq1$.

Proof.

(1) $$\left|\int_0^x\int_0^s (|\Re_+(\varphi_n(t)+\varphi_m(t))|^2-|\Re_+(2\psi)|^2)\,dtds\right|$$

$$\leqq\left|\int_0^x\int_0^s (|\Re_+\varphi_n(t)+\Re_+\varphi_m(t)|^2-4|\Re_+\psi|^2)\,dtds\right|$$

$$\leqq\left|\int_0^x\int_0^s(|\mathfrak{R}_+\varphi_n(t)|^2+|\mathfrak{R}_+\varphi_m(t)|^2+2|\mathfrak{R}_+\varphi_n(t)|\cdot|\mathfrak{R}_+\varphi_m(t)| -4|\mathfrak{R}_+\psi|^2)\,dtds\right|$$

$$\leqq\left|\int_0^x\int_0^s(2|\mathfrak{R}_+\varphi_n(t)|^2+2|\mathfrak{R}_+\varphi_m(t)|^2-4|\mathfrak{R}_+\psi|^2)\,dtds\right|$$

$$<4\varepsilon.$$

We can prove similarly inequalities. So $\varphi_n+\varphi_m\in U_{4\varepsilon}(2\psi)$.

(2) $$\left|\int_0^x\int_0^s(|\mathfrak{R}_+\cdot\alpha\varphi_n(t)|^2-|\mathfrak{R}_+\cdot\alpha\psi(t)|^2\,dtds\right|\leq|\alpha|^2\varepsilon.$$

We can prove other inequalities also. So $\alpha\varphi_n\in U_{|\alpha|^2\varepsilon}(0)$.

(3) $$\left|\int_0^x\int_0^s\{|\mathfrak{R}_+(\alpha\varphi_n+\beta\varphi_m)|^2-|\mathfrak{R}_+\psi(t)|^2\}\,dtds\right|$$

$$\leq|\alpha|^2\left|\int_0^x\int_0^s(|\mathfrak{R}_+\varphi_n|^2-|\mathfrak{R}_+\psi(t)|^2)\,dtds\right|+$$

$$+|\beta|^2\left|\int_0^x\int_0^s(|\mathfrak{R}_+\varphi_m|^2-|\mathfrak{R}_+\psi|^2)\,dtds\right|+$$

$$+2|\alpha\cdot\beta|\left|\int_0^x\int_0^s(|\mathfrak{R}_+\varphi_n|\cdot|\mathfrak{R}_+\varphi_m|-|\mathfrak{R}_+\psi(t)|^2)\,dtds\right|$$

$$<|\alpha|^2\varepsilon+|\beta|^2\varepsilon+2|\alpha\cdot\beta|$$

$$\cdot\left|\int_0^x\int_0^s((|\mathfrak{R}_+\varphi_n|^2+|\mathfrak{R}_+\varphi_m|^2)/2-|\mathfrak{R}_+\psi(t)|^2)\,dtds\right|$$

$$<(|\alpha|^2+|\beta|^2+2|\alpha\cdot\beta|)\varepsilon=(\alpha+\beta)^2\varepsilon=\varepsilon.$$

We can prove similarly other inequalities. Hence $\alpha\varphi_n+\beta\psi_n\in U_\varepsilon(0)$ for $\alpha+\beta=1$, $0\leqq\alpha\leqq1$, $0\leqq\beta\leqq1$. From this lemma we conclude that the space $L^2[0,1]$ is a convex topological space.

But it is not linear. Because, we can construct the following example;

We can define $\sqrt{\delta(\frac{1}{2})}$, using above remark. Now let $\Psi_1=\{\varphi_{1n}\}\in\overline{L^2}[0,1]$, $\Psi_2=\{\varphi_{2n}\}\in\overline{L^2}[0,1]$ be the sequences which satisfy the following conditions $\Psi_1=\Psi_2=\sqrt{\delta(1/2)}$, $\varphi_{1n}>0$, $\varphi_{2n}>0$ and Carrier $(\varphi_{1n})\cap$Carrier $(\varphi_{2n})=\phi$. Then $\{\varphi_{1n}+\varphi_{2n}\}=\sqrt{2\delta(1/2)}\neq2\sqrt{\delta(1/2)}$.

4. Unified representation of state vectors. In order to construct the linear topological space which represents the space of state vectors, we select subclasses from the space $\overline{L^2}[0,1]$ as follows:

At the first step, from any equivalent class $(\{\varphi_n\})\in\overline{L^2}[0,1]$ we select a particular sequence $\{\varphi_n^0\}$ as follows:

For $\Psi\in L^2[0,1]$, we define $\varphi_n^0=\psi$ i.e. $\Psi=\{\psi,\psi,\cdots\}$.

For $\psi\in\overline{L^2[0,1]}-L^2[0,1]$, we select $\{\varphi_n^0\}$ by the following way.

Let Δ_{n,x_0} denote the following functions;

$$\Delta_{n,x_0}\begin{cases} =\sqrt{\rho_{n,x_0}(x)} & \text{for } x\in[0,1] \\ =0 & \text{for } x\notin[0,1], \text{ where } x_0\in[0,1]. \end{cases}$$

For $\psi=\sqrt{\delta(x_0)}$ $(x_0\neq 0,1)$, we define $\{\varphi_n^0\}=\{\Delta_{n,x_0}\}$.

For $\psi=\sqrt{\delta(x_0)}$ $(x_0=0,1)$, we define $\{\varphi_n^0\}=\{\sqrt{2}\,\Delta_{n,x_0}\}$.

For $\psi=e^{i\theta}\sqrt{\delta(x_0)}$ $(x_0\neq 0,1)$, we define $\{\varphi_n^0\}=\{e^{i\theta}\Delta_{n,x_0}\}$.

For $\psi=e^{i\theta}\sqrt{\delta(1)}$ and $e^{i\theta}\sqrt{\delta(0)}$, we define $\{\varphi_n^0\}=\{e^{i\theta}\sqrt{2}\,\Delta_{n,x_0}\}$.

By these selections, from every equivalent class of $\overline{\boldsymbol{L}^2}[0,1]$, we can select $\{\varphi_n^0\}$ because the following lemma holds.

Lemma 4. *In $\overline{\boldsymbol{L}^2}[0,1]$, there is no other element than the countable linear combinations of the following elements;*

(1) $\psi_j\in L^2[0,1]$

(2) $\sqrt{\delta_{x_0}(x)}$ $(0\leqq x\leqq 1)$.

Proof. Using the above stated remark, we see that

if $\langle (f_n^{\mathfrak{R}+})^2,\varphi\rangle\geqq 0$ for all $0<\varphi\in(\boldsymbol{D})$, then $\langle T^{\mathfrak{R}+},\varphi\rangle\geqq 0$ for all $0<\varphi\in(\boldsymbol{D})$,

if $\langle (f_n^{\mathfrak{R}-})^2,\varphi\rangle\geqq 0$ for all $0<\varphi\in(\boldsymbol{D})$, then $\langle T^{\mathfrak{R}-},\varphi\rangle\geqq 0$ for all $0<\varphi\in(\boldsymbol{D})$,

if $\langle (f_n^{\mathfrak{I}+})^2,\varphi\rangle\geqq 0$ for all $0<\varphi\in(\boldsymbol{D})$, then $\langle T^{\mathfrak{I}+},\varphi\rangle\geqq 0$ for all $0<\varphi\in(\boldsymbol{D})$,

if $\langle (f_n^{\mathfrak{I}-})^2,\varphi\rangle\geqq 0$ for all $0<\varphi\in(\boldsymbol{D})$, then $\langle T^{\mathfrak{I}-},\varphi\rangle\geqq 0$ for all $0<\varphi\in(\boldsymbol{D})$.

From these results $T^{\mathfrak{R}+}$, $T^{\mathfrak{R}-}$, $T^{\mathfrak{I}+}$, $T^{\mathfrak{I}-}$ are positive measures. Hence we can obtain the conclusion of this lemma [1].

In the 2nd step, we construct set of subclasses which satisfy the uniformly equivalent condition and include the above selected particular sequence, where uniformly equivalence is defined as follows:

Definition. *If there exists $\varepsilon_n>0$ such that $\lim_{n\to\infty}\varepsilon_n=0$ and $|\varphi_n-\psi_n|<\varepsilon_n$ then we say that $\{\varphi_n\}$ is* **uniformly equivalent** *to $\{\psi_n\}$, and denote it by $\{\varphi_n\}\cong\{\psi_n\}$.*

Using these notations, uniformly equivalent subclass is expressed as follows:

$$[\{\varphi_n^0\}]=\left\{\{\varphi_n\};\ \{\varphi_n\}\in(\{\varphi_n^0\}),\ \{\varphi_n\}\cong\{\varphi_n^0\}\right\}$$

Lemma 5. *If $\{\varphi_n\}$ and $\{\psi_n\}$ are Cauchy sequence and if $\{\varphi_n\}\cong\{\tilde{\varphi}_n\}$, $\{\psi_n\}\cong\{\tilde{\psi}_n\}$, then $\{\varphi_n+\psi_n\}\cong\{\tilde{\varphi}_n+\tilde{\psi}_n\}$, $\{\alpha\varphi_n\}\cong\{\alpha\tilde{\varphi}_n\}$, and $\{\tilde{\varphi}_n\}$, $\{\tilde{\psi}_n\}$, $\{\varphi_n+\psi_n\}$, $\{\alpha\varphi_n\}$ are Cauchy sequences.*

Let's define the inner product $\langle\{\tilde{\varphi}_n\},\{\tilde{\psi}_n\}\rangle=\lim_{n\to\infty}\int_0^1\varphi_n\tilde{\psi}_n\,dt$.

Lemma 6. *If $\{\tilde{\varphi}_n\}\cong\{\varphi_n\}$, $\{\tilde{\psi}_n\}\cong\{\psi_n\}$, then $\langle\{\tilde{\varphi}_n\},\{\tilde{\psi}_n\}\rangle=\langle\{\varphi_n\},\{\psi_n\}\rangle$.*
Let's define $\langle[\{\varphi_n\}],[\psi_n\}]\rangle$ by $\langle\{\varphi_n^0\},\{\psi_n^0\}\rangle$ where $\{\varphi_n^0\in\}[\{\varphi_n\}],\{\psi_n^0\}\in[\{\psi_n\}]$. Let's call the space L of uniformly equivalent classes which is constructed in the above 2nd step a space of generalized state vectors.

Now our state vector's space L must satisfy the following Theorem;

Theorem. *A space of generalized state vectors satisfy the conditions* (**1**) *and* (**2**).

(1) *in the space L, we can define an innerproduct,*
(2) *L is a complete linear topological space.*

References

[1] L. Schwartz: Theorie des distributions. I, II, Hermann (1950), (1951).

[2] И. М. Гельфанд и Г. Е. Шилов: Оьобщенные Функции и Действия над Ними. I. II. III (1959).

[3] R. Haag: Canonical communition relations in Field theory and functional Intergration. Lectures in Theoretical Physics, Vol. **III** (1960), Interscience publishers in, New York.

[4] L. Garding and A. S. Wightman: Representations of the anticommutation relation, Representations of the commutation relation, Proc. Natl. Acad. Sci. U.S.A., **40**, 617 (1956).

[5] I. E. Segal: Foundation of the theory of dynamical systems of infinitely many degrees of freedom, Danske. Videnskab. Selskab Mat.-fys. **31**, Nr. 12 (1959).

[6] K. O. Friedrichs: Methematical Aspects of the Quantum Theory of Field. Interscience, New York (1953).

[7] T. Ishihara and H. Yamagata: On distributions and spaces of sequences. I. General theory, Proc. Japan Acad., **38**, 327–332 (1962).

[8] F. Hausdorff: Mengenlehre. Walter de Gruger, Berlin (1935).

[9] T. Ishihara and H. Yamagata: On distributions and spaces of sequences. III (to appear).