70. A Remark on the Concept of Channels

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1. In the mathematical theory of information (cf. [4]), a message is represented by a sequence of alphabets. Considerable messages are represented by the points of the infinite product space of alphabets which is called the message space associated with a measure on the field of all measurable subsets generated by the cylinder sets. A *channel* $[X, \nu, Y]$ is represented by a function $\nu(x, E)$ where x runs through an input message space X and E varies all measurable subsets of an output message space Y, which is usually assumed to satisfy the following two conditions:

(i) For almost all x, $\nu(x, E)$ is a probability measure on Y, and

(ii) for all E, $\nu(x, E)$ is a measurable function of x.

If $[X, \nu, Y]$ is a channel and if μ is a probability measure on X, then

$$\lambda(E)=\int_X \nu(x, E)d\mu(x) \tag{1}$$

defines a probability measure λ on Y. Being defined by

$$\lambda=K_*\mu, \tag{2}$$

K_* maps the space of all probability measures on X into that of Y.

In the above description, it is not essential, as recently observed by H. Umegaki, that the measure space X is the direct product of alphabets. He discussed the case that the output and input spaces are simply measure or measurable spaces. Again, in his case, (2) defines a linear mapping K_* which carries the probability measures into the probability measures.

Being considered probability measures as normal states of the abelian von Neumann algebras of all bounded measurable functions on the output and input spaces, Umegaki's discussion suggests us to generalize the concept of channels for not necessarily commutative von Neumann algebras which will be given a preliminary discussion in the below.

2. Basing on the definitions of Dixmier [2], let us suppose that A_* and B_* are the subconjugate space of all ultraweakly continuous linear functionals on von Neumann algebras A and B respectively. A positive linear transformation K_* defined on A_* with its range in B_* is called a *generalized channel* provided that K_* preserves the norm of positive elements:

$$||K_*\rho||=||\rho||, \tag{3}$$

for every $\rho \geqq 0$. Since a normal state σ of A is a positive elements of A_* with the norm unity, (3) is equivalent to state that K_* *maps a normal state into a normal state.* Also, since it is well known that $\|\sigma\| = \sigma(1)$ for a positive linear functional σ, (3) implies that $K_*\sigma(1) = \|K_*\sigma\| = \|\sigma\| = \sigma(1)$. Let K be the conjugate linear transformation of K_* which maps B into A, then

$$\rho(Kb) = K_*\rho(b), \tag{4}$$

for every $\rho \in A_*$. Since the set of all normal states is total, $K_*\sigma(1) = \sigma(1)$ for all normal states implies $K1 = 1$. This proves the half of the following

PROPOSITION 1. *A positive linear transformation K_* on A_* into B_* is a generalized channel if and only if the conjugate transformation K is a positive normal transformation on B into A preserving the identity.*

If K is a normal positive linear transformation preserving the identity, then there is a positive linear transformation K_* satisfying $(K_*)^* = K$ by a theorem of Dixmier [2; Chap. I, §4, Thm. 2], which satisfies $K_*\sigma(1) = \sigma(K1) = \sigma(1)$ for every positive σ of A_*, whence K_* maps normal states into normal states.

According to Proposition 1, a positive normal transformation K among von Neumann algebras is called also a *generalized channel* provided that K preserves the identity. Hence, a normal homomorphism is a generalized channel whenever it preserves the identity.

3. In the usual signal observation, it is natural to suppose that a state of the output space corresponds to some states of the input space or there is no redundant state for the receiver. Hence, in our non-commutative case, it is also natural to expect that a generalized channel K is *faithful* provided that K_* maps the space of all normal states of A onto that of B. Since a theorem of Takeda [6] and Grothendieck [3] shows that B_* is generated by the linear combinations of normal states of B, a faithful generalized channel K_* maps A_* onto B_*.

PROPOSITION 2. *If a generalized channel K is faithful then K maps B into A faithfully in the usual sense.*

If K is a faithful generalized channel, then K_* maps A_* onto B_* as noted above, whence K^{-1} is continuous by a theorem of Banach [1; 148], this shows that K is one-to-one.

The converse of Propostion 2 is true if K^{-1} is positive and bounded. This is the case when K is an isomorphism onto a von Neumann subalgebra, which is a part of a theorem of Sakai [5].

References

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