# 69. Projective Limits and Metric Spaces with u-Extension Properties

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A metric space is said to *have a u-extension property* if any uniformly continuous real map defined on any subspace can always be extended uniformly over the whole space. Corson and Isbell [6] proved the theorem that a metric space has a u-extension property if and only if its completion is a projective limit [5] of fine metric spaces. We know [1, 3] some conditions characterizing a metric space with a $u$-extension property. Using the conditions and applying the idea of Flachsmeyer [7], we are, in this note, going to prove the same theorem with a somewhat simpler projective system.

We know (Theorem 2, [1]) that *a metric complete space $S$ has a u-extension property if and only if, for any natural number $n$, there is a compact subset $K_n$ such that for any open set $G$ containing $K_n$ there is a natural number $m$ satisfying $V^{\infty}_{1/m}(x)\subset V_{1/n}(x)$ for every point $x\notin G$*, where $V_{1/n}$ is the entourage $\{(x, y); d(x, y)<1/n\}$ of the uniform structure of the space and $V^{\infty}_{1/m}(x)$ is the set of all points which are joined with $x$ by $V_{1/m}$-chains.

$K_n$ in this statement is taken as the set of all points $x$ satisfying $V^{\infty}_{1/i}(x)\not\subset V_{1/n}(x)$ for any $i$ [3]. For each $x\notin K_n$, we take the least natural number $i(n, x)$ of numbers $j$ with $V^{\infty}_{1/j}(x)\subset V_{1/n}(x)$, and put

$$H_n(x)=V^{\infty}_{1/i(n,x)}(x).$$

(1) *$H_m(y)\supset H_n(x)$ if and only if $H_m(y)\cap H_n(x)\neq\phi$ and $i(m, y)\leq i(n, x)$.*

In fact, if $H_m(y)\supset H_n(x)$ and $i(m, y)>i(n, x)$, then $H_n(x)\supset V^{\infty}_{1/i(n,x)}(y)$, and so $V^{\infty}_{1/i(n,x)}(y)=V^{\infty}_{1/i(m,y)}(y)$, which contradicts the definition of $i(m, y)$.

Hence there is the greatest $H_n(y)$ containing $H_n(x)$ whose $i(n, y)$ is the least of $i(n, z)$ with $H_n(z)\supset H_n(x)$, such the $H_n(y)$ is denoted by $G_n(x)$.

(2) *$G_n(x)\neq G_n(y)$ implies $G_n(x)\cap G_n(y)=\phi$.*

We put

$$J_n=K_n-\bigcup_{x\notin K_n} G_n(x)$$

and have the equivalent relation $R_n$ on $S$ defined by the cover

$$\alpha_n=\{(p), G_n(x); p\in J_n, x\in S-K_n\},$$

where $(p)$ is the singleton, namely, $xR_n\, y$ if no member of $\alpha_n$ includes only one of the points $x$ and $y$ [7].

(3) *$\alpha_{2n}$ refines $\alpha_n$.*

In fact, if $x\in S-K_{2n}$, then $x\in S-K_n$. We assume $G_n(x)=H_n(x_1)$, $G_{2n}(x)=H_{2n}(x_2)$ and $i(n,x_1)>i(2n,x_2)$, then $H_n(x_1)\subset H_{2n}(x_2)$ and $d(x_2,x_1)<1/2n$. On the other hand, $H_n(x_1)\subset V^{\infty}_{1/i(2n,x_2)}(x_1)=V^{\infty}_{1/i(2n,x_2)}(x_2)$, so we have $V^{\infty}_{1/i(2n,x_2)}(x_1)\subset V_{1/n}(x_1)$, which contradicts the definition of $i(n,x_1)$. Hence we have $i(n,x_1)\leq i(2n,x_2)$ and $G_n(x)\supset G_{2n}(x)$ by (1).

(4) *$xR_{2n}y$ implies $xR_ny$.* Therefore, we can now write $R_n\leq R_{2n}$ (cf. [7]).

We define the distance function $\delta(u,v)$ of the points $u$ and $v$ in the set $S/R_n$ by $\delta(u,v)=d(u',v')$ which is the distance between the inverse images $u'$ and $v'$ in $S$ of $u$ and $v$ by the canonical map $\varphi_n$ on $S$ to $S/R_n$.

(5) *$\delta$ is compatible with the topology of the quotient space $S/R_n$.*

In fact, let $\mathfrak{T}$ be the quotient topology and $U$ an open neighborhood in $\mathfrak{T}$ of a point $u$ of $S/R_n$, then $U'=\varphi_n^{-1}(U)$ is open in $S$. (i) When $u'\cap J_n=\phi$, $u'=\varphi_n^{-1}(u)$, then there is $x\in S-K_n$ such that $u'=G_n(x)$, and we have $V_{1/i(n,x)}(u')=u'$, $V_{1/i(n,x)}(u)=(u)$. (ii) When $u'\cap J_n\neq\phi$, then $u'=x\in J_n$, and we have $V_{1/m}(x)\subset U'$ for some $m$, so $V_{1/m}(u)\subset U$ because $\varphi_n^{-1}(v)\cap U'\neq\phi$, $v\in S/R_n$, implies $\varphi_n^{-1}(v)\subset U'$. Conversely, since $\bigcup\{v';\delta(u,v)<1/m\}$ is open in $S$, $\{v;\delta(u,v)<1/m\}$ is open in $\mathfrak{T}$.

(6) *$\{R_{2^n};n=1,2\cdots\}$ is fundamental* [7], namely, all open sets in $S$ which are saturated with respect to the relations build a basis of open sets in $S$, and no two different points in $S$ are equivalent to each other with respect to all the relations.

In fact, let $E$ be an open set in $S$ including a point $x$, then we have $V_{1/n}(x)\subset E$ for some $n$. When $x\in J_{4n}$, then we have $E\supset\bigcup\{u';u\in\alpha_{4n},\delta(x,u)<1/4n\}$. When $x\notin J_{4n}$, then $G_{4n}(x)\subset E$. Moreover, if $d(x,y)>1/n$, then $x\overline{R}_{2n}y$ because $\operatorname{dia}G_{2n}(z)<1/n$ for any $z\in S$.

(7) Consequently, when we write $f_{n,2n}$ for the canonical map of $S/R_{2n}$ to $S/R_n$, which maps an $R_{2n}$-class to the $R_n$-class containing the $R_{2n}$-class, then it is uniformly continuous and we have the projective system [5] $(S/R_{2^n},f_{2^m,2^n};m,n=1,2,\cdots)$ of metric spaces and the projective limit $S^*=\underleftarrow{\lim}\,S/R_{2^n}$ which contains $S$ as a dense subspace by identifying $x\in S$ with $(\varphi_{2^n}(x))$ (Satz 1, [7]).

(8) *$S/R_n$ is fine [6] for every $n$.*

In fact, let us suppose that $\{u_1,u_2,\cdots\}$, $u_i\in S/R_n$, does not have any accumulation point. We take a point $x_i\in u_i'=\varphi_n^{-1}(u_i)$ for every $i$, then $\{x_i\}$ does not have any accumulation point in $S$. Therefore, the number of $u_i'$ meeting the compact $K_n$, say $u_1',\cdots,u_r'$, is finite, so $A=\bigcup_{i>r}u_i'$ is closed and disjoint from $K_n$. There is $k$ such that $V^{\infty}_{1/k}(x)$

$\subset V_{1/n}(x)$ for all $x \in A$ (cf. Theorem 2 in [1] cited at the first part of this note), and so $V_{1/k}(G_n(x)) = G_n(x)$ for all $x \in A$, i.e. $V_{1/k}(u_i) = u_i$, $i > r$, and hence $S/R_n$ is fine (Theorem 1, [4]).

(9) *$S$ is a uniform subspace of $S^*$.*

In fact, let $f_n$ be the canonical map of $S^*$ to $S/R_n$, and put $g_n = f_n \times f_n$, then we have $g_{5m}^{-1}(\{(u, v); u, v \in S/R_{5m}, \delta(u, v) < 1/5m\}) \cap (S \times S) \subset \{(x, y); x, y \in S, d(x, y) < 1/m\}$, and $\{(x, y); d(x, y) < 1/m\} \subset g_n^{-1}(\{(u, v); u, v \in S/R_n, \delta(u, v) < 1/m\}) \cap (S \times S)$ for any $m$ and $n$.

Therefore, we have $S^* = S$ by the completeness of $S$, and, from Corollaries 1, 2 in [2] and Theorem 1.4 in [6], which is an immediate consequence from Theorem[1] in [2], we have

**Theorem** (Corson and Isbell [6]). *A metric space has a u-extension property if and only if its completion is a projective limit of fine metric spaces.*

**Remark.** The proof of Corollary 1 in [2] is not correct. Though we can readily prove it in the same direction, we shall here show a simple proof in another way.

Let $\{A_n\}$ be a $U$-discrete sequence of subsets, $\{a_n\}$ a sequence of natural numbers, and $V^2 \subset U$. There is a continuous real map $f$ on $S$ with the value $a_n$ on $A_n$ and 0 on $S\text{-}\bigcup V(A_n)$. Since $S$ is *uc*, $f$ is uniformly continuous, so $S$ has a $u$-extension property by the lemma and by the theorem stated before the corollary.

## References


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1) In the proof of the "if" part of the theorem in [2], $f$ should read as non-negative (we may assume it without loss of generality); the same is true for $n$.