65. On Bertrand's Problem in an Arithmetic Progression

By Tikao TATUZAWA
Mathematical Institute, Nagoya University, Nagoya
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In this note, we shall prove the following

Theorem. There exists a positive constant c such that, if
$$x \geqq \exp(c \log k \log\log k)$$
and k is sufficiently large, then
$$\pi(2x; k, l) - \pi(x; k, l) > 0$$
is true for all l, satisfying $(k, l) = 1$.

We shall use the same notations and symbols as in Prachar's book [Primzahlverteilung, Springer, 1957].

If $x \geqq \exp(k)$, the theorem is true by Theorem 8.3 in p. 144 or Theorem 3.2 in p. 323 of the book. Hence, we assume that

(1) $$\exp(c \log k \log\log k) \leqq x \leqq \exp(k).$$

Consequently,

(2) $$c \log k \log\log k \leqq \log x, \quad \frac{c}{2} \log\log x \leqq \frac{\log x}{\log k},$$

if k is sufficiently large.

We know from the results of Page [see IV, §5 and §6] and Linnik [see X, §3] that there exists a positive constant c_0 such that there are no zeros of any L-function mod k in the rectangle
$$1 - \frac{c_0}{\log k} \leqq \sigma \leqq 1, \quad |t| \leqq k^4$$
except possible one real zero β_1 of a particular L-function formed with a real character. Further if we put
$$\delta_0 = \begin{cases} 1-\beta_1 & \text{if the exceptional zero exists,} \\ \dfrac{c_0}{\log k} & \text{otherwise,} \end{cases}$$
then the rectangle
$$1 - \lambda(k) \leqq \sigma \leqq 1, \quad |t| \leqq k^4$$
contains no zero of any L-function mod k except β_1, where

(3) $$\lambda(k) = \frac{c_0}{\log k} \log \frac{c_0 e}{\delta_0 \log k}.$$

Now the constant c in the theorem will be given such that

(4) $$cc_0 \geqq 20.$$

Proof. From p. 321 of Prachar's book, we obtain
$$\varphi(k)\{\psi(2x; k, l) - \psi(x; k, l)\}$$

$$\geqq x\left\{1-E_1x^{\beta_1-1}-O\left(\sum_{\chi}\sum_{|\gamma|\leqq T}{}' x^{\beta-1}+\frac{\varphi(k)\log^2 x}{T}+\frac{\log x}{x^{3/4}}\right)\right\},$$

for $x\geqq T\geqq 2$. Putting $T=k^4$ and using (3.6) in p. 322 of the book, we deduce

$$\geqq x\left\{1-E_1x^{\beta_1-1}-O\left(\frac{k^6}{x}+\left(\frac{k^{13}}{x}\right)^{\lambda(k)}\log^8 k\log x+\frac{\log^2 x}{k^3}+\frac{\log x}{x^{3/4}}\right)\right\}.$$

It follows from (1), (2), (3), and (4) that

$$E_1x^{\beta_1-1}+c_1\left(\frac{k^{13}}{x}\right)^{\lambda(k)}\log^8 k\log x$$

$$\leqq x^{-\delta_0}+c_1\left(\frac{c_0e}{\delta_0\log k}\right)^{\frac{c_0}{\log k}(13\log k-\log x)}\frac{1}{c^8}\log^9 x$$

$$\leqq e^{-\delta_0\log x}+c_1\left(\frac{c_0e}{\delta_0\log k}\right)^{13c_0-\frac{cc_0}{2}\log\log x}\frac{1}{c^8}\log^9 x$$

$$\leqq e^{-\delta_0 c\log k\log\log k}+\frac{c_1}{c^8}\left(\frac{c_0e}{\delta_0\log k}\right)^{13c_0-\log\log x}.$$

We consider now two cases.

(i) $$\delta_0\, c\log k\log\log k\leqq 1.$$

Since $e^{-z}\leqq 1-\frac{1}{2}z$ for $0\leqq z\leqq 1$ and $e\leqq\frac{c_0e}{\delta_0\log k}$,

$$E_1x^{\beta_1-1}+c_1\left(\frac{k^{13}}{x}\right)^{\lambda(k)}\log^8 k\cdot\log x$$

$$\leqq 1-\frac{1}{2}\delta_0\, c\log k\log\log k+\frac{c_1}{c^8}\frac{\delta_0\log k}{c_0e}e^{13c_0+1-\log\log x}$$

$$\leqq 1-\frac{1}{2}\delta_0\, c\log k\log\log k+\frac{c_1e^{13c_0}}{c_0c^8}\frac{\delta_0\log k}{\log x}.$$

Hence, noting that $x\leqq\exp(k)$, we have

$$\varphi(k)\{\psi(2x;k,l)-\psi(x;k,l)\}$$

$$\geqq x\left\{\frac{c}{2}\delta_0\log k\log\log k-\frac{c_1e^{13c_0}}{c_0c^8}\delta_0\frac{\log k}{\log x}-O\left(\frac{k^6}{x}+\frac{1}{k}+\frac{\log x}{x^{3/4}}\right)\right\},$$

which is positive if k is sufficiently large, with the aid of Siegel's result [Theorem 8.2, p. 144 in the book].

(ii) $$\delta_0\, c\log k\log\log k\geqq 1.$$

In this case,

$$E_1x^{\beta_1-1}+c_1\left(\frac{k^{13}}{x}\right)^{\lambda(k)}\log^8 k\log x$$

$$\leqq e^{-1}+\frac{c_1}{c^8}e^{13c_0-\log\log x}=e^{-1}+\frac{c_1e^{13c_0}}{c^8}\frac{1}{\log x}\leqq c_2<1$$

if k is sufficiently large. Hence, noting that $x\leqq\exp(k)$, we have

$$\varphi(k)\{\psi(2x;k,l)-\psi(x;k,l)\}$$

$$\geqq x\left\{1-c_2-O\left(\frac{k^6}{x}+\frac{1}{k}+\frac{\log x}{x^{3/4}}\right)\right\},$$

which is positive if k is sufficiently large.

Thus we get the desired result.