112. On an Isomorphism of Galois Cohomology Groups $H^m(G, O_K)$ of Integers in an Algebraic Number Field

By Hideo YOKOI

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Introduction. In my paper [1][1] we proved the following theorem:

Let K be a normal extension over the rational number field $\boldsymbol{Q}$, and k be a subfield of K such that K/k is cyclic of prime degree p and that $k/\boldsymbol{Q}$ is normal of degree n. Then, for every dimension m the Galois cohomology group $H^m(G, O_K)$ $(G=G(K/k))$ of O_K with respect to K/k is isomorphic to the ns/e-ple direct sum of cyclic group of order p:

$$H^m(G, O_K)\cong\{\overbrace{p, p, \cdots, p}^{ns/e}\}.$$

There we proved this Theorem by showing that the 1-dimensional Galois cohomology group $H^1(G, O_K)$ of O_K with respect to K/k is isomorphic to the 0-dimensional Galois cohomology group $H^0(G, O_K)$ of O_K with respect to K/k:

$$H^1(G, O_K)\cong H^0(G, O_K).$$

In the present paper, we shall give another proof of this Theorem by showing that the 0-dimensional Galois cohomology group $H^0(G, O_K)$ of O_K with respect to K/k is isomorphic to the -1-dimensional Galois cohomology group $H^{-1}(G, O_K)$ of O_K with respect to K/k:

$$H^0(G, O_K)\cong H^{-1}(G, O_K).$$

Theorem. *Let K be a normal extension over the rational number field $\boldsymbol{Q}$, and k be a subfield of K such that K/k is cyclic of prime degree p and that $k/\boldsymbol{Q}$ is normal of degree n. Denote by G the Galois group of K/k, by O_K and O_k the rings of all integers in K and k respectively. Further, let v be the common ramification number with respect to K/k of all the prime divisors $\mathfrak{P}_i$ of p in K,[2] and e be the common ramification order with respect to $k/\boldsymbol{Q}$ of all the prime divisors $\mathfrak{p}_i$ of p in k. Put $s=v-\left[\frac{v}{p}\right]\geqq 0$, where $[x]$ means Gaussian symbol.*

Then the -1-dimensional Galois cohomology group $H^{-1}(G, O_K)$ of O_K with respect to K/k is isomorphic to the ns/e-ple direct sum of cyclic group of order p:

1) Cf. H. Yokoi [1], Theorem 3.

2) Here we understand the ramification number v in the same way as we understood in [1]. Cf. H. Yokoi [1], [Remark].

$$H^{-1}(G, O_K)\cong\{\overbrace{p, p, \cdots, p}^{ns/e}\}.$$

Proof. Let O_K^* be the submodule of O_K which consists of all elements A in O_K such that $S_{K/k}A=0$, σ be a generator of Galois group G of K/k, and $Z[G]$ be the group ring of G over the rational integers. Set $S=1+\sigma+\cdots+\sigma^{p-1}\in Z[G]$. Then we may regard O_K^* as a $Z[G]/(S)$-module, where (S) is the principal ideal of $Z[G]$ generated by S. Since $Z[G]/(S)$ is a Dedekindian ring, we have by Chevalley's lemma[3] direct decompositions

$$(\#)\qquad \begin{aligned} O_K^*&=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_n,\\ (\sigma-1)O_K&=\mathfrak{B}_1\oplus\mathfrak{B}_2\oplus\cdots\oplus\mathfrak{B}_n\end{aligned}$$

of $Z[G]/(S)$-modules O_K^* and $(\sigma-1)O_K$ with $\mathfrak{A}_i\supseteqq\mathfrak{B}_i$ $(i=1,2,\cdots,n)$. Since $(\sigma-1)O_K^*$ is also a $Z[G]/(S)$-submodule of $(\sigma-1)O_K$, corresponding to the decompositions $(\#)$ we have the direct decomposition

$$(\sigma-1)O_K^*=(\sigma-1)\mathfrak{A}_1\oplus\cdots\oplus(\sigma-1)\mathfrak{A}_n$$

of $(\sigma-1)O_K^*$ such that $\mathfrak{A}_i\supseteqq\mathfrak{B}_i\supseteqq(\sigma-1)\mathfrak{A}_i$ $(i=1,2,\cdots,n)$. Further, since the index of $(\sigma-1)\mathfrak{A}_i$ in $\mathfrak{A}_i$ is the prime number p, each factor $\mathfrak{B}_i$ is equal to either $\mathfrak{A}_i$ or $(\sigma-1)\mathfrak{A}_i$.

On the other hand, if we put $t=ns/e$, we may take the following basis of the ring O_K:[4]

$$O_K=[\beta_1,\cdots,\beta_{n(p-1)},\ \omega_1,\cdots,\omega_t,\ \xi_1,\cdots,\xi_{n-t}]$$

with $\xi_i=(\omega_{t+i}+\alpha_i)/p$ $(i=1,2,\cdots,n-t)$, where $[\omega_j\ (j=1,2,\cdots,n)]$ is a suitable basis of O_K, and α_i, β_m are integers in K such that $S_{K/k}\alpha_i=0$, $S_{K/k}\beta_m=0$, $(i=1,2,\cdots,n-t,\ m=1,2,\cdots,n(p-1))$. Then both $(\sigma-1)O_K$ and $(\sigma-1)O_K^*$ are submodules of O_K^* generated by $(\sigma-1)\beta_1,\cdots,(\sigma-1)\beta_{n(p-1)}$, $(\sigma-1)\xi_1,\cdots,(\sigma-1)\xi_{n-t}$ and $(\sigma-1)\beta_1,\cdots,(\sigma-1)\beta_{n(p-1)}$ respectively. Therefore, $(\sigma-1)\xi_i=(\sigma-1)\alpha_i/p$ $(i=1,2,\cdots,n-t)$ generate the factor module $F=(\sigma-1)O_K/(\sigma-1)O_K^*$ and moreover they form a basis of F. For, if we assume that $\Sigma_i x_i(\sigma-1)\xi_i=(\sigma-1)\beta$ for some β in O_K^* and for rational integers x_i $(i=1,2,\cdots,n-t)$, then we have $(\sigma-1)(\Sigma_i x_i\alpha_i/p-\beta)=0$ and hence $p\beta=\Sigma_i x_i\alpha_i$. But since $\alpha_i=p\xi_i-\omega_i$, we have $\Sigma_i x_i\omega_i=p(\beta-\Sigma_i x_i\xi_i)$, which implies $x_i\equiv0 \bmod p$ for every $i=1,2,\cdots,n-t$.

This fact means, as we see by permuting the summands in $(\#)$, that $\mathfrak{B}_1=(\sigma-1)\mathfrak{A}_1$, $\mathfrak{B}_2=(\sigma-1)\mathfrak{A}_2,\cdots,\mathfrak{B}_t=(\sigma-1)\mathfrak{A}_t$ and $\mathfrak{B}_{t+1}=\mathfrak{A}_{t+1},\cdots,\mathfrak{B}_n=\mathfrak{A}_n$. Consequently, we have the following direct decompositions:

$$F=(\sigma-1)O_K/(\sigma-1)O_K^*\cong\mathfrak{A}_{t+1}/(\sigma-1)\mathfrak{A}_{t+1}\oplus\cdots\oplus\mathfrak{A}_n/(\sigma-1)\mathfrak{A}_n,$$
$$O_K^*/(\sigma-1)O_K\cong\mathfrak{A}_1/(\sigma-1)\mathfrak{A}_1\oplus\cdots\oplus\mathfrak{A}_t/(\sigma-1)\mathfrak{A}_t.$$

This implies our isomorphism

$$H^{-1}(G, O_K)\cong O_K^*/(\sigma-1)O_K\cong\{\overbrace{p, p, \cdots, p}^{t}\}. \qquad \text{Q.E.D.}$$

From this Theorem and Proposition 6 in [1], we can prove the

3) Cf. C. Chevalley [3].

4) Cf. H. Yokoi [2], Theorem 2.

isomorphism between the -1-dimensional Galois cohomology group $H^{-1}(G, O_K)$ of O_K with respect to K/k and the 0-dimensional Galois cohomology group $H^0(G, O_K)$ of O_K with respect to K/k in the same way as in [1] we proved the isomorphism between the 1-dimensional Galois cohomology group $H^1(G, O_K)$ of O_K with respect to K/k and the 0-dimensional Galois cohomology group $H^0(G, O_K)$ of O_K with respect to K/k. Consequently, we obtain the isomorphism of all dimensional Galois cohomology $\mathrm{H}^m(G, O_K)$ of O_K with respect to K/k again similarly.

References

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[2] ——: On the ring of integers in an algebraic number field as a representation module of Galois group, Nagoya Math. J., **16**, 83–90 (1960).

[3] C. Chevalley: L'arithmétique dans les algèbres des matrices, Actual. Sci. Ind., **323**.