111. A Note on the Extension of Semigroups with Operators

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In this note we shall report some theorems concerning the theory of extension of semigroups with operators, without detailed proof. By a mono-endomorphism of a semigroup we mean a one-to-one endomorphism of a semigroup. Let S be a semigroup which is not necessarily commutative and suppose that Γ is a commutative semigroup of some mono-endomorphisms α of S, that is, Γ is not necessarily composed of all mono-endomorphisms of S. Let $+$ denote the operation in S and αx the image of an element x under α.

$$\left.\begin{array}{ll}(1.1) & \alpha(x+y)=\alpha x+\alpha y \\ (1.2) & (\alpha\beta)x=(\beta\alpha)x \\ (1.3) & \alpha x=\alpha y \text{ implies } x=y\end{array}\right\} \text{ for } \alpha, \beta\in\Gamma;\ x, y\in S.$$

We shall call such an S a semigroup with Γ denoted by (s, Γ).

Theorem 1. *For (S, Γ), there exists $(\bar{S}, \bar{\Gamma})$ such that*

(2.1) *S is embedded into $\bar{S}$,*

(2.2) *Γ and $\bar{\Gamma}$ are isomorphic,*

(2.3) *Each $\bar{\alpha}\in\bar{\Gamma}$ is an extension of $\alpha\in\Gamma$ to $\bar{S}$, and $\bar{\alpha}$ is an automorphism of $\bar{S}$.*

(2.4) *$(\bar{S}. \bar{\Gamma})$ is the smallest extension of (S, Γ) in the following meaning: If $(\bar{\bar{S}}, \bar{\bar{\Gamma}})$ is any extension satisfying* (2.1), (2.2), *and* (2.3), *then $\bar{S}$ is embedded into $\bar{\bar{S}}$.*

Proof. Consider the set of all pairs (x, α) of $x\in S$ and $\alpha\in\Gamma$ and we introduce a relation as $(x, \alpha)\sim(y, \beta)$ iff $\beta x=\alpha y$. Then it is an equivalence relation. Let $\overline{(x, \alpha)}$ denote an equivalence class containing (x, α) and let $\bar{S}$ be the set of all equivalence classes. We define an operation in $\bar{S}$ as follows:

$$\overline{(x, \alpha)}+\overline{(y, \beta)}=\overline{(\beta x+\alpha y, \alpha\beta)}.$$

It is shown that this operation is single valued on $\bar{S}$, and $\bar{S}$ is a semigroup into which S is embedded under the mapping $\Sigma: S\ni x\rightarrow \overline{(\alpha x, \alpha)}\in\bar{S}$ where $\overline{(\alpha x, \alpha)}$ is independent of the choice of α. For each α, a mapping $\bar{\alpha}$ of $\bar{S}$ into $\bar{S}$ is defined as follows:

$$\bar{\alpha}\overline{(z, \gamma)}=\overline{(\alpha z, \gamma)}.$$

We can see that this mapping is single-valued on $\bar{S}$ and $\bar{\alpha}$ is a mono-endomorphism of $\bar{S}$. Clearly the mapping $\alpha\to\bar{\alpha}$ gives an isomorphism of Γ to $\bar{\Gamma}$. It follows from the definition of $\bar{\alpha}$ that $\bar{\alpha}$ is an extension of α. In the proof of automorphism, we must show that $\bar{\alpha}$ is subjective. In fact, for any $\overline{(x, \alpha)}\in\bar{S}$ and any $\bar{\gamma}\in\bar{\Gamma}$

$$\bar{\gamma}\overline{(x, \gamma\alpha)}=\overline{(\gamma x, \gamma\alpha)}=\overline{(x, \alpha)}.$$

Finally, to prove (2.4), let $\bar{\bar{\alpha}}$ be an extension of α to $\bar{\bar{S}}$. If we define the mapping T of $\bar{S}$ into $\bar{\bar{S}}$ as

$$T\overline{(x, \alpha)}=y \quad \text{where } \bar{\bar{\alpha}}y=\Sigma x, x\in S, y\in\bar{\bar{S}},$$

then we can prove that T is an isomorphism of $\bar{S}$ into $\bar{\bar{S}}$.

By the way, if $\bar{S}$ is cancellative, then $\bar{S}$ is also; if S is a group, so is $\bar{S}$; if S is commutative $\bar{S}$ is also.

Since $\bar{\Gamma}$ in Theorem 1 is commutative and cancellative, it is possible to embed $\bar{\Gamma}$ into a group.

Theorem 2. *For* (S, Γ), *there exists* $(\bar{S}, \Gamma^*)$ *such that*

(3.1) *S is embedded into $\bar{S}$,*

(3.2) *Γ^* is the smallest commutative group into which Γ is embedded,*

(3.3) *Each $\beta\in\Gamma^*$ is an automorphism of $\bar{S}$. If $\alpha\in\Gamma$ is mapped to $\alpha^*\in\Gamma^*$ under the embedding of Γ into Γ^*, then each α^* is an extension of $\alpha\in\Gamma$ to $\bar{S}$.*

(3.4) *If $(\bar{\bar{S}}, \Gamma^{**})$ is any extension satisfyiny* (3.1), (3.2), *and* (3.3), *then $\bar{S}$ and Γ^* are embedded into $\bar{\bar{S}}$ and Γ^{**} respectively.*

Proof. By Theorem 1, we have obtained an extension $(\bar{S}, \bar{\Gamma})$ of (S, Γ). Consider the set Γ^* of all pairs $((\bar{\alpha}, \bar{\beta}))$ of elements of $\bar{\Gamma}$ with identifying $((\bar{\alpha}, \bar{\beta}))=((\bar{\gamma}, \bar{\delta}))$ as $\bar{\delta}\bar{\alpha}=\bar{\beta}\bar{\gamma}$. To simplify the notations, $((\bar{\alpha}, \bar{\beta}))$ denotes again the equivalence class containing $((\bar{\alpha}, \bar{\beta}))$. We define a mapping $((\bar{\alpha}, \bar{\beta}))$ of $\bar{S}$ into itself as follows

$$((\bar{\alpha}, \bar{\beta}))\overline{(z, \gamma)}=\overline{(\alpha z, \beta\gamma)}$$

clearly $$((\bar{\alpha}, \bar{\beta}))((\bar{\gamma}, \bar{\delta}))=((\overline{\alpha\gamma}, \overline{\beta\delta}))=((\bar{\alpha}\bar{\gamma}, \bar{\beta}\bar{\delta})).$$

It is shown that $((\bar{\alpha}, \bar{\beta}))$ is an automorphism of $\bar{S}$ and $\bar{\Gamma}$ is embedded into Γ^* with the correspondence

$$\bar{\alpha}\to((\bar{\gamma}\bar{\alpha}, \bar{\gamma}))=\alpha^*$$

where α^* is easily seen to be an extension of α to $\bar{S}$. As is well known, Γ^* is the smallest group containing Γ.

Remark. Instead of Γ, consider Γ_0 as follows: Suppose

(4.1) Γ_0 contains a zero-mapping ζ, i.e., a mapping of all elements to a definite element.

(4.2) ζ is a two-sided zero of Γ_0.

(4.3) Γ_0 contains no zero-divisor.

Then we get the similar theorems such as Theorems 1, 2.

Let S be a commutative semigroup. For every positive integer n, we consider an endomorphism $\boldsymbol{n}$ of S:

$$\boldsymbol{n}\cdot x=\underbrace{x+\cdots+x}_{n}.$$

The operator semigroup Γ of some endomorphisms of this kind is considered as a subsemigroup of the multiplicative semigroup of positive integers. A commutative semigroup T is said to be uniquely Γ-divisible if for any $x\in T$ and for any $\boldsymbol{n}\in\Gamma$, there is exactly one $y\in T$ such that $\boldsymbol{n}\cdot y=x$. S is said to be Γ-cancellative if $\boldsymbol{n}\cdot x=\boldsymbol{n}\cdot y$ implies $x=y$ for every $\boldsymbol{n}\in\Gamma$.

As an application of Theorem 1, we get immediately

Theorem 3. *If a commutative semigroup S is Γ-cancellative then S is embedded in the smallest uniquely Γ-divisible semigroup.*

We define a semiring R to be an algebraic system with two binary operations—addition and multiplication—such that for every $x, y, z\in R$

(5.1) $(x+y)+z=x+(y+z)$

(5.2) $(xy)z=x(yz)$

(5.3) $x(y+z)=xy+xz, \qquad (y+z)x=yx+zx.$

Theorem 4. *If the multiplicative semigroup of a semiring R is commutative and if, for any non-zero element a,*

$$ab=ac \text{ implies } b=c,$$

then R is embedded into the smallest semiring R^* such that the multiplicative semigroup of R^* is commutative group or group with zero.

Let S be a commutative semigroup and let Γ be a multiplicative semigroup of positive integers and suppose S is cancellative and Γ-cancellative, that is,

$$x+y=x+z \quad \text{implies } y=z$$
$$\boldsymbol{n}\cdot x=\boldsymbol{n}\cdot y \quad \text{implies } x=y \quad \text{for every } n\in\Gamma.$$

$S^{\mathfrak{g}}$ denotes the smallest group containing S and $S^{\mathfrak{d}}$ the smallest uniquely Γ-divisible semigroup containing S. Then we have

$$(S^{\mathfrak{g}})^{\mathfrak{d}}\cong(S^{\mathfrak{d}})^{\mathfrak{g}},\ (S^{\mathfrak{g}})^{\mathfrak{g}}\cong S^{\mathfrak{g}},\ (S^{\mathfrak{d}})^{\mathfrak{d}}\cong S^{\mathfrak{d}}$$

where $\cong$ means "isomorphic".

If we regard these results as $\mathfrak{g}\mathfrak{d}=\mathfrak{d}\mathfrak{g}, \mathfrak{g}^2=\mathfrak{g}, \mathfrak{d}^2=\mathfrak{d}$, then it follows that $\mathfrak{g}$ and $\mathfrak{d}$ generate a semilattice of order 3. Further applying these operations to direct product, we have

$$(S_1\times S_2)^{\mathfrak{g}}\cong S_1^{\mathfrak{g}}\times S_2^{\mathfrak{g}},\ (S_1\times S_2)^{\mathfrak{d}}=S_1^{\mathfrak{d}}\times S_2^{\mathfrak{g}}$$

where both S_1 and S_2 are cancellative and Γ-cancellative.

Added note: We have not used associativity of S in the proof of Theorems 1, 2 except for the proof of associativity of $\bar{S}$. Therefore the theorems are available for the extension $(\bar{S}, \bar{\Gamma})$ or $(\bar{S}, \Gamma^*)$ of a groupoid (S, Γ) with operators, where a groupoid is a system with a binary operation and the conditions concerning Γ are not changed.