108. Some Characterizations of $\mathfrak{m}$-paracompact Spaces. I

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Recently K. Morita [4] has introduced a notion of $\mathfrak{m}$-paracompactness, and proved some interesting results concerning it. For any infinite cardinal number $\mathfrak{m}$, a topological space X is said to be $\mathfrak{m}$-paracompact if any open covering of X with power $\leqq \mathfrak{m}$ (i.e. consisting of at most $\mathfrak{m}$ sets) admits a locally finite open covering as its refinement.

The purpose of this paper is to give some characterizations of $\mathfrak{m}$-paracompactness, which are related to the results obtained by H. H. Corson [1] and E. Michael [3] to characterize paracompactness of a topological space.

1. The following theorem is a modification of Corson's result ([1, Theorem 1]).

Theorem 1. *For a normal space X the following statements are equivalent.*

(a) *X is $\mathfrak{m}$-paracompact.*

(b) *If $\mathfrak{F}$ is a filter base on X with power $\leqq \mathfrak{m}$ (i.e. consisting of at most $\mathfrak{m}$ sets) such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped, then $\mathfrak{F}$ has a cluster point in X.*

Proof. (a)$\rightarrow$(b). Let $\mathfrak{F}=\{F_\lambda \mid \lambda \in \Lambda\}$ be a filter base in X with $|\Lambda| \leqq \mathfrak{m}$ such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped.[1] Assume that $\mathfrak{F}$ has no cluster point in X. Then $\mathfrak{G}=\{X-\bar{F}_\lambda \mid \lambda \in \Lambda\}$ is an open covering of X with power $\leqq \mathfrak{m}$, since $\bigcap \bar{F}_\lambda=\phi$. Let $\mathfrak{U}=\{U_\alpha \mid \alpha \in \Omega\}$ be a locally finite open refinement of $\mathfrak{G}$, where we can assume that $|\Omega| \leqq \mathfrak{m}$. Since X is normal, there exists a closed covering $\{K_\alpha \mid \alpha \in \Omega\}$ of X such that $K_\alpha \subset U_\alpha$ for every $\alpha \in \Omega$. Hence there exists, for each α, a continuous function f_α: $X \rightarrow I=[0, 1]$ such that $f_\alpha(x)$ is 1 or 0 according as $x \in K_\alpha$ or $x \in X-U_\alpha$. Now for every point x of X we assign an element $\varphi(x)=\{f_\alpha(x) \mid \alpha \in \Omega\}$. Let $Y=\varphi(X)$, and let us introduce a distance d in Y such that

$$d(\varphi(x_1), \varphi(x_2))=\sum_{\alpha \in \Omega} |f_\alpha(x_1)-f_\alpha(x_2)|,$$

where $\varphi(x_i)=\{f_\alpha(x_i) \mid \alpha \in \Omega\}$ $(i=1, 2)$. Then it is obvious that φ is a continuous mapping of X onto a metric space Y. Let $V_\alpha=\{\varphi(x) | f_\alpha(x)>0\}$.

1) For any set A, we denote by $|A|$ the cardinal number of A.

Then $\{V_\alpha | \alpha \in \Omega\}$ is an open covering of Y such that $\varphi^{-1}(V_\alpha) \subset U_\alpha$. We must now prove that any point y of Y is not a cluster point of $\{\varphi(F_\lambda) | \lambda \in \Lambda\}$. Since y is contained in some V_α, and $\varphi^{-1}(V_\alpha) \subset U_\alpha \subset X - \overline{F}_\lambda$ for some $\lambda \in \Lambda$, we obtain $\varphi^{-1}(V_\alpha) \cap F_\lambda = \phi$. Hence $V_\alpha \cap \varphi(F_\lambda) = \phi$. Thus $\{\varphi(F_\lambda) | \lambda \in \Lambda\}$ has no cluster point in Y.

(b)$\rightarrow$(a). Let $\mathfrak{U} = \{U_\lambda | \lambda \in \Lambda\}$ be an open covering of X with power $\leqq \mathfrak{m}$. Then $\mathfrak{F} = \{F_\lambda | \lambda \in \Lambda\}$ is a filter base of power $\leqq \mathfrak{m}$, where $F_\lambda = X - U_\lambda$. Let Y be any metric space such that X is continuously mapped into it, and we denote by f this continuous mapping. Under the assumption that $\mathfrak{U}$ has no locally finite open refinement, we shall prove that the image of $\mathfrak{F}$ has a cluster point in Y, that is, $\bigcap \overline{f(F_\lambda)} \neq \phi$. Then we have $\bigcap F_\lambda \neq \phi$ by (b), which contradicts $\bigcap F_\lambda = \phi$. Now let $\bigcap \overline{f(F_\lambda)} = \phi$. Then $\mathfrak{G} = \{G_\lambda | \lambda \in \Lambda\}$ is an open covering of Y, where $G_\lambda = Y - \overline{f(F_\lambda)}$. Since Y is a metric space, there exists a locally finite normal open refinement $\mathfrak{H} = \{H_\alpha | \alpha \in \Omega\}$ of $\mathfrak{G}$. Then $\{f^{-1}(H_\alpha) | \alpha \in \Omega\}$ is a locally finite normal open covering of X. Since $\mathfrak{U}$ has no locally finite open refinement, there exists a set $f^{-1}(H_\alpha)$ such that $f^{-1}(H_\alpha) \not\subset U_\lambda$ for every $\lambda \in \Lambda$. Therefore $H_\alpha \cap f(F_\lambda) \neq \phi$ for every $\lambda \in \Lambda$. This is contradictory to the fact that $G_\lambda \cap \overline{f(F_\lambda)} \neq \phi$ for every $\lambda \in \Lambda$, because H_α is contained in some $G_\lambda \in \mathfrak{G}$. Hence we have $\bigcap \overline{f(F_\lambda)} \neq \phi$. This completes the proof.

In the proof that (b)$\rightarrow$(a), we do not use the assumption that X is normal. Therefore we have the following

Corollary 1. ([1, Theorem 1]) *For a Hausdorff space X the following statements are equivalent.*

(a) *X is paracompact.*

(b) *If $\mathfrak{F}$ is a filter in X such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped, then $\mathfrak{F}$ has a cluster point in X.*

If $\mathfrak{m} = \aleph_0$ in Theorem 1, we have the following

Corollary 2. *For a normal space X the following statements are equivalent.*

(a) *X is countably paracompact.*

(b) *If $\mathfrak{F}$ is a countable filter base in X such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped, then $\mathfrak{F}$ has a cluster point in X.*

2. The following theorem is a modification of a theorem of E. Michael ([3, Proposition 2]) and is essentially proved by K. Morita [4]. We shall give here our proof based on the same idea as in the proof of [3, Proposition 2].

Theorem 2. *The following properties of a normal space are equivalent.*

(a) *X is $\mathfrak{m}$-paracompact.*

(b) *Every open covering of X with power $\leqq \mathfrak{m}$ has a partition of unity subordinated to it.*

Proof. (a)$\to$(b). This is trivial, since every open covering of X with power $\leqq \mathfrak{m}$ has a locally finite partition of unity.

(b)$\to$(a). As a first step, we prove that X is countably paracompact. For this purpose, by [2, Theorem 3], it is sufficient to prove that each countable open covering of X admits a σ-locally finite closed refinement. But we can prove that each open covering of X with power $\leqq \mathfrak{m}$ admits a σ-locally finite closed refinement. In fact, let $\mathfrak{U}=\{U_\lambda \mid \lambda \in \Lambda\}$ be an open covering of X with $|\Lambda| \leqq \mathfrak{m}$. Then there exists a partition of unity Φ subordinated to it. For each positive integer i, let $\mathfrak{F}_i$ be the collection of all sets of the form $\{x \in X \mid \phi(x) \geqq 1/i\}$, with $\phi \in \Phi$, and let $\mathfrak{F}=\bigcup_{i=1}^{\infty} \mathfrak{F}_i$. Clearly $\mathfrak{F}$ is a closed refinement of $\mathfrak{U}$. To prove that $\mathfrak{F}$ is locally finite, pick a finite subset Φ_0 of Φ, for any $x_0 \in X$, such that $\sum_{\phi \in \Phi_0} \phi(x_0) > 1-1/2i$, and then pick a neighborhood W of x_0 such that $\sum_{\phi \in \Phi_0} \phi(x) > 1-1/i$ for all $x \in W$. Then W cannot intersect $\{x \in X \mid \phi(x) \geqq 1/i\}$ unless $\phi \in \Phi_0$, and therefore W intersects only finitely many elements of $\mathfrak{F}_i$. Hence $\mathfrak{F}$ is a σ-locally finite closed refinement of $\mathfrak{U}$. Therefore X is countably paracompact. By the similar arguments as above we can show that every open covering of X with power $\leqq \mathfrak{m}$ admits a σ-locally finite open refinement. (Replace $\{x \in X \mid \phi(x) \geqq 1/i\}$ with $\{x \in X \mid \phi(x) > 1/i\}$.) Hence, by [4, Theorem 1.1 (*e*)], X becomes $\mathfrak{m}$-paracompact. This completes the proof.

It should be noted that, in the proof that (b)$\to$(a), we do not use normality of X. For a topological space X, as the proof above shows, (b) implies that every open covering of X with power $\leqq \mathfrak{m}$ admits a σ-locally finite open refinement. For a T_1-space X, (b) implies also complete regularity of X. By using these properties, Michael [3] has proved his result ([3, Proposition 2]):

The following properties of a T_1-space are equivalent.

(a) *X is paracompact.*

(b) *Every open covering of X has a partition of unity subordinated to it.*

Corollary. *The following properties of a normal space are equivalent.*

(a) *X is countably paracompact.*

(b) *Every countable open covering of X has a partition of unity subordinated to it.*

References

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