104. Relations among Topologies on Riemann Surfaces. III

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Proposition 3. $\{p_n^1\}$ *and* $\{p_n^2\}$ *determine the same K-Martin's point relative to* ${}_l\mathfrak{D}_\infty$ *for every* l.

Domain ${}_l\Omega$ and Ω. Put $\mathfrak{D}^*=\Re-\tilde{s}_0-\sum_{n=1}^{\infty}(s_n^1+s_n^2+s_n^3+\tilde{s}_n+R_n-\Lambda_n)$ $(={}_1\mathfrak{D}_\infty)$. Let Γ'_n be a simply connected domain containing R_n such that $\partial\Gamma'_n$ intersects Λ_n such that

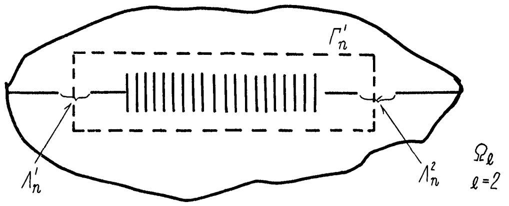
$$\Gamma'_n:\alpha-0.75\leqq Re\ z\leqq\alpha+0.75,\ \frac{6}{2^n}-\frac{6}{2^{n+3}}\leqq Im\ z\leqq\frac{6}{2^n}-\frac{6}{2^{n+3}},$$

where $\alpha=1.5$ or 4.5 according as n is odd or even.

Let T_n be a system of vertical segments in R_n such that

$$T_n=\sum_{i=0}^{k}t_n^i,$$

$$t_n^i:Re\ z=\alpha+\frac{i}{k},\ \frac{6}{2^n}-\frac{6}{2^{n+4}}\leqq Im\ z\leqq\frac{6}{2^n}+\frac{6}{2^{n+4}},$$



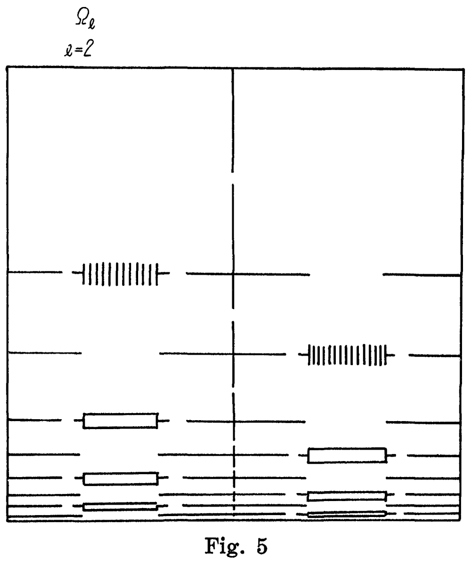


Fig. 5

where $\alpha=1$ or 4 according as n is odd or even and $i=0,1,2,\cdots k$.

Let $G(z, z_0, \mathfrak{D}^*)$ be the Green's function of $\mathfrak{D}^*$. Put $N_n=\min G(z, z_0, \mathfrak{D}^*)$ on $\partial\Delta_0'+\partial\Gamma_n'$ as z_0 varies in Δ_0. Then $N_n>0$. Let $G^{T_n}(z, z_0, \mathfrak{R})$ and $G^{R_n}(z, z_0, \mathfrak{R})$ be Green's functions of $\mathfrak{R}-T_n$ and $\mathfrak{R}-R_n$ respectively. Since $G^{T_n}(z, z_0, \mathfrak{R})\to G^{R_n}(z, z_0, \mathfrak{R})$ uniformly on $\partial\Delta_0'+\partial\Gamma_n'$ independent of z_0 as $k(n)\to\infty$, there exists a number $k(n)$ such that

$$G^{T_n}(z, z_0, \mathfrak{R})-G^{R_n}(z, z_0, \mathfrak{R})\leqq\frac{1}{4^n}G(z, z_0, \mathfrak{D}^*) \text{ on } \partial\Delta_0'+\partial\Gamma_n' \text{ for any } z_0\in\Delta_0. \quad (12)$$

We suppose T_n is defined as above.

Put ${}_l\Omega=\mathfrak{R}-\tilde{s}_0-\sum_1^\infty(s_n^1+s_n^2+s_n^3+\tilde{s}_n+R_n-\Lambda_n)+\sum_1^l(R_n-T_n)$ and $\Omega=\lim_l {}_l\Omega$. Then ${}_l\mathfrak{D}_\infty-{}_l\Omega$ is compact in ${}_l\mathfrak{D}_\infty$. Hence by proposition 3 and by lemma 1 we have

Proposition 4. *$\{p_n^1\}$ and $\{p_n^2\}$ determine the same K-Martin's point relative to ${}_l\Omega$ for every l.*

Now $\Omega-{}_l\Omega=\sum_{l+1}^\infty(R_n-T_n)$. We shall show that $\{p_n^1\}$ and $\{p_n^2\}$ determine the same K-Martin's point relative to Ω.

We denote by $G^A(z, z_0, B)$ the Green's function of $B-A$. By $T_n\subset\partial\Omega$, $G^{T_n}(z, z_0, \Omega)=G^{\Sigma T_n}(z, z_0, \Omega)=G(z, z_0, \Omega)$. We have by $D^*\subset\Omega\subset\mathfrak{R}$ and by (12) and by lemma 4 (putting $T_n=F_2\subset R_n=F_1$, $0=E_2\subset E_1=\mathfrak{R}-\Omega$) $G^{T_n}(z, z_0, \Omega)-G^{R_n}(z, z_0, \Omega)\leqq G^{T_n}(z, z_0, \mathfrak{R})-G^{R_n}(z, z_0, \mathfrak{R})\leqq\frac{1}{4^n}G(z, z_0, D^*)\leqq\frac{1}{4^n}G(z, z_0, \Omega)$ on $\partial\Delta_0'+\partial\Gamma_n'$ for any $z_0\in\Delta_0$. Now $G^{T_n}(z, z_0, \Omega)-G^{R_n}(z, z_0, \Omega)=0=G(z, z_0, \Omega)$ on $\partial\Omega_n-\Gamma_n'$ and $G^{T_n}(z, z_0, \Omega)-G^{R_n}(z, z_0, \Omega)\leqq\frac{1}{4^n}G(z, z_0, \Omega)$ on $\partial\Delta_0'+\partial\Gamma_n'$. Hence by the maximum principle $G^{T_n}(z, z_0, \Omega)-G^{R_n}(z, z_0, \Omega)\leqq\frac{1}{4^n}G(z, z_0, \Omega)$ in $\Omega-\Delta_0'-\Gamma_n'$. Thus

$$G^{\sum_{l+1} T_n}(z, z_0, \Omega)-G^{\sum_{l+1} R_n}(z, z_0, \Omega)\leqq\sum_{l+1}^\infty(G^{T_n}(z, z_0, \Omega)-G^{R_n}(z, z_0, \Omega))$$
$$\leqq\sum_{l+1}^\infty\frac{1}{4^n}G(z, z_0, \Omega) \qquad \text{in } \Omega-\Delta_0'-\sum\Gamma_n'.$$

Now $\Omega-\sum_{l+1}^\infty R_n={}_l\Omega$ and $G^{\sum_{l+1} R_n}(z, z_0, \Omega)=G(z, z_0, {}_l\Omega)$, whence

$$G(z, z_0, \Omega)-G(z, z_0, \Omega_l)\leqq\sum_{l+1}^\infty\frac{1}{4^n}G(z, z_0, \Omega) \quad \text{in } \Omega-\sum_{l+1}^\infty\Gamma_n'-\Delta_0'$$

for any $z_0\in\Delta_0$.

On the other hand, $p_n^i\in\Omega-\sum^\infty\Gamma_n'-\Delta_0'$ and $\Omega_l\uparrow\Omega$. Hence by Proposition 4 and by Lemma 5 $\{p_n^1\}$ *and* $\{p_n^2\}$ *determine the same K-Martin's point relative to* Ω.

We shall show that there exist subsequences $\{p_{n'}^1\}$ and $\{p_{n'}^2\}$ of $\{p_n^1\}$ and $\{p_n^2\}$ which determine different N-Martin's points relative to

Ω. Put $\Omega'=\Omega-\Delta_0$. Let $N(z,p)$ be an N-Green's function of Ω' such that $N(z,p)=0$ on $\partial\Delta_0$, $N(z,p)$ has a positive logarithmic singularity at p and has minimal $\overset{*}{\text{D}}$irichlet integral. Let $U(z)$ be a Dirichlet bounded harmonic function in Ω' such that $U(z)$ has minimal Dirichlet integral and $U(z)=-1$ on $\partial\Delta_0^1$ and $U(z)=1$ on $\partial\Delta_0^2$. Then

$$U(p_n^i)=\frac{1}{2\pi}\int_{\partial\Delta_0} U(z)\frac{\partial}{\partial n}N(z,p_n^i)ds. \tag{13}$$

Put $C_n: C_n=E\left[z:\left|z-3-3\left(\frac{1}{2^n}+\frac{1}{2^{n-1}}\right)i\right|<\frac{6}{2^{n+2}}\right]$ and let C_n' be a circle with the same centre as C_n with radius$=a_n:\log\frac{(6/2^{n+2})}{a_n}=m_n$. Let $U'(z)$ be a continuous function in Ω' such that $U(z)=-1$ in $E[z:z\in\Omega', Re\, z<3]-\sum_1^\infty C_n$, $U(z)=1$ in $E[z:z\in\Omega', Re\, z>3]-\sum_1^\infty C_n$ and $U'(z)$ is harmonic in $\sum_{n=1}^\infty(C_n-C_n')$ and $U'(z)=0$ in $\sum C_n'$. Then $D(U'(z))=\sum_{n=1}^\infty D_{C_n}(U'(z))=2\pi\sum\left(\frac{1}{m_n}\right)<\frac{1}{32}$. Hence by the Dirichlet principle $D(U(z))\leqq D(U'(z))<\frac{1}{32}$. Consider the behaviour of $U(z)$ on the domain:

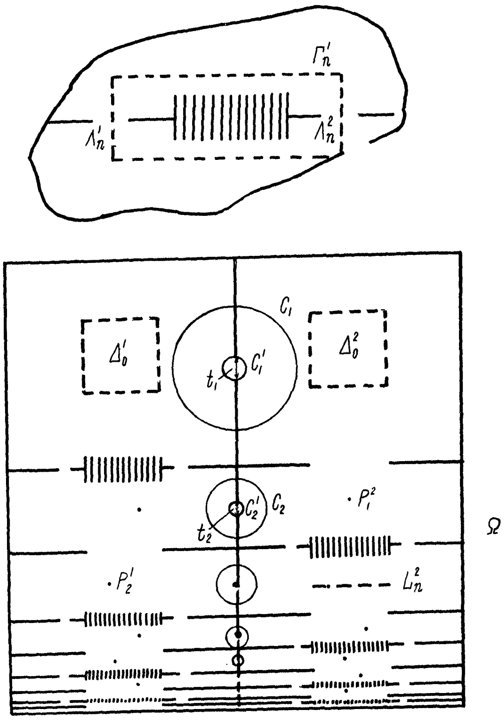


Fig. 6

$4 < Re\, z < 5$, $4 > Im\, z > \frac{1}{2}\left(\frac{6}{2^n} + \frac{6}{2^{n+1}}\right) = y_n$. Then by Lemma 2, $D(U(z)) \geqq \int |U(x+iy_n) - U(x+4i)|^2 ds$. Put $L_n^2 = E[z: 5 < Re\, z < 6,\ Im\, z = y_n]$ and $L_n^1 = E[z: 1 < Re\, z < 2,\ Im\, z = y_n]$. Assume the measure of $E\left[z \in L_n^2: U(z) \geqq \frac{1}{2}\right]$ is larger than $\frac{1}{2}$. Then $\frac{1}{32} \geqq D(U(z)) \geqq \left(\frac{1}{4}\right) \times \frac{1}{2} = \frac{1}{16}$. This is a contradiction. Hence there exists a set $\overset{*}{L}{}_n^2$ in L_n^2 of positive measure $\left(> \frac{1}{2}\right)$ in which $U(z) > \frac{1}{2}$. Choose a point p_n^2 in $\overset{*}{L}{}_n^2$ and also choose a subsequence $p_{n'}^2$ from $\{p_n^2\}$ such that $\{p_{n'}^2\}$ determine an N-Martin's point of Ω'. Then $\varliminf_{n'} U(p_{n'}^2) \geqq \frac{1}{2}$. Similarly we can choose $p_{n'}^1$ in L_n^1 such that $\varlimsup_{n'} U(p_{n'}^1) \leqq -\frac{1}{2}$ and a subsequence $\{p_{n'}^1\}$ determining an N-Martin's point. Assume $\{p_{n'}^1\}$ and $\{p_{n'}^2\}$ determine the same point. Then by (13) $\lim_{n'} U(p_{n'}^1) = \lim_{n'} U(p_{n'}^2)$. This is a contradiction. Hence $\{p_{n'}^1\}$ and $\{p_{n'}^2\}$ determine different N-Martin's points relative to Ω. But $\{p_{n'}^i\}$ is a subsequence of $\{p_{n'}^i\}$, whence $\{p_{n'}^1\}$ and $\{p_{n'}^2\}$ determine the same K-Martin's point relative to Ω. *Thus* $KM.T \not\succ NM.T$.

We shall show $NM.T \succ KM.T$. Y. Toki[1] constructed a Riemann surface R with following properties: 1°). R is a covering surface over $|z| < 1$. 2°) R is obtained by connecting infinitely many leaves which are identical to the unit circle. 3°) $R \subset O_{HD}$ and $R \not\subset O_{AB}$. We see easily that every boundary point of R is regular for the Green's function. Hence by Theorem 16[2] every boundary point of R with respect to N-Martin's topology is singular of second kind (if the harmonic measure of a point p is positive, we call p a singular point of second kind). Hence also by the same theorem there exists only one N-Martin's boundary point. On the other hand, $O_{AB} \not\ni R$ implies that R has no singular K-Martin's boundary point (if the harmonic measure of a point is positive, we call it singular) and R has infinitely many K-Martin's boundary points. This example shows $NM.T \not\succ KM.T$. But it is more interesting to show $NM.T \not\succ KM.T$ by an example of a Riemann surface of planer character.

Lemma 7. *Let R be a Riemann surface and let G be its subdomain. Let $\{v_n\}$ be a decreasing sequence of domain such that $\bigcap v_n = 0$. Let $U(z)(V(z))$ be a positive harmonic function in $R(G)$*

1) Y. Tôki: On the examples in the classification of open Riemann surfaces, Osaka Math. J., **5** (1953).

2) Z. Kuramochi: Singular points of Riemann surfaces, Journ. Hokkaido Univ., (1962).

such that $U(z)(V(z))$ *is the least positive harmonic function in* $R - v_n(G-v_n)$ *larger than* $U(z)(V(z))$ *on* ∂v_n. *Let* ${}^{v_n}_{inex}U(z)({}^{v_n}_{ex}V(z))$ *be the positive least harmonic function in* $G-v_n(R-v_n)$ *larger than* $U(z)(V(z))$ *on* ∂v_n. Then ${}^{v_n}_{inex}U(z)\downarrow({}^{v_n}_{ex}V(z)\uparrow)$. *We denote this limit by* ${}_{inex}U(z)$ *(from* R *to* G *relative to* v_n*)* $({}_{ex}V(z)$ (from G to R relative to v_n)). *Then*

$$\text{if } {}_{ex}V(z)<\infty,\ V(z)={}_{inex}({}_{ex}V(z)).^{3)} \tag{14}$$

Let G_1 *and* G_2 *be domains such that* $G_1\cap G_2=0$ *and* $V^i(z)$ *be a harmonic function in* G_i *with* $V^i(z)=0$ *on* ∂G_i. *Then* ${}_{ex}V^1(z)(<\infty)$ *and* ${}_{ex}V^2(z)(<\infty)$ *are linearly independent.*[3]

Lemma 8. *Let* R *and* G *be those of Lemma* 7. *Let* p_0 *be a fixed point in* G *and let* $\{p^i_n\}$ *be a sequence such that* $K(z, p^i_n, G)$ $\left(=\dfrac{G(z, p^i_n, G)}{G(p_0, p^i_n, G)}\right)$ *and* $K(z, p^i_n, R)\left(\dfrac{G(z, p^i_n, R)}{G(p_0, p^i_n, R)}\right)$ *converge to* $K(z, \{p^i_n\}, G)$ *and* $K(z, \{p^i_n\}, R)$. *Let* $\{v_n\}$ *be a decreasing domains such that* $v_n \ni p^i_n, p^i_{n+1}\cdots$ *and* $\bigcap v_n=0$. *If there exists a constant* M *such that* $G(p_n, p^i_0, R)<MG(p_n, p^i_0, G)$ *for* $n\geqq n_0$, *then* ${}_{ex}K(z, \{p^i_n\}, G)<\infty$. *Suppose* $K(z, \{p^1_n\}, G)$ *and* $K(z, \{p^2_n\}, G)$ *are linearly independent. Then by* (14) *we see at once* ${}_{ex}K(z, \{p^1_n\}, G)$ *and* ${}_{ex}K(z, \{p^2_n\}, G)$ *are linearly independent.*

Now $K(z, p_n, R)=\dfrac{G(z, p_n, R)}{G(p_0, p_n, R)}\geqq\dfrac{G(z, p_n, G)}{MG(p_0, p_n, G)}=\dfrac{K(z, p_n, G)}{M}$. Since $K(z, p_n, R)$ is positive in R and $K(z, p_n, R)>\dfrac{K(z, p_n, G)}{M}$ for $n\geqq n_0$, $\infty>K(z,\{p_n\},R)>\dfrac{{}_{ex}K(z, \{p_n\}, G)}{M}$. Hence $\infty>K(z,\{p_n\},R)>\dfrac{{}_{ex}K(z, \{p_n\}, G)}{M}$.

3) Z. Kuramochi: Relations between harmonic dimensions, Proc. Japan Acad., (1954).