102. *Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. II*

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Let $\{\lambda_\nu\}$, $S(\lambda)$, $\Phi(\lambda)$, and $R(\lambda)$ be the same notations as those defined in the statement of Theorem 1 [3] respectively, and $\Psi(\lambda)$ the second principal part of $S(\lambda)$ in the case where all the accumulation points of $\{\lambda_\nu\}$ form an uncountable set.

Since, by Theorem 1,

$$\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{S(\lambda)}{(\lambda-z)^{k+1}}d\lambda=\frac{R^{(k)}(z)}{k!}\quad(k=0,1,2,3\cdots)$$

for every point z in the interior of the circle $|\lambda|=\rho$ with $\sup_\nu|\lambda_\nu|<\rho<\infty$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{S(\rho e^{it})}{(\rho e^{it})^k}dt=\frac{R^{(k)}(0)}{k!}.$$

Consequently $R(\lambda)$ is expansible, on the domain $\{\lambda:|\lambda|<\infty\}$, in terms of integrals concerning the given function $S(\lambda)$ itself.

In this paper I have mainly two purposes: one is to find the expressions of $\Phi(\lambda)$ and $\Psi(\lambda)$ in terms of integrals concerning $S(\lambda)$ itself respectively, the other is to establish the relation between the maximum-modulus of $S(\lambda)$ on the circle $|\lambda-c|=\rho_1$ containing $\{\lambda_\nu\}$ and all the accumulation points of $\{\lambda_\nu\}$ inside itself and that of $R(\lambda)$ on the circle $|\lambda-c|=\rho_2$ with $\rho_2<\rho_1$.

Theorem 4. If the set of all the accumulation points of $\{\lambda_\nu\}$ is uncountable, then the second principal part $\Psi(\lambda)$ of $S(\lambda)$ in Theorem 1 is expressible in the form

$$(1)\qquad \Psi\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt-\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{e^{it}}{e^{it}-\kappa e^{i\theta}}dt-\sum_{\alpha=1}^{m}\sum_\nu c_\alpha^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}-\lambda_\nu\right)^{-\alpha}$$

for every κ with $0<\kappa<1$ and every ρ with $\sup_\nu|\lambda_\nu|<\rho<\infty$; and if, contrary to this, the set of all the accumulation points of $\{\lambda_\nu\}$ is countable, then

$$(2)\qquad \sum_{\alpha=1}^{m}\sum_\nu c_\alpha^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}-\lambda_\nu\right)^{-\alpha}=\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt-\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{e^{it}}{e^{it}-\kappa e^{i\theta}}dt$$

for such κ, ρ as above.

Proof. It first follows from Theorem 1 that for every point z on the domain $\{z : |z| < \rho\}$ with $\sup_{\nu} |\lambda_\nu| < \rho < \infty$

$$\begin{aligned} R(z) &= \frac{1}{2\pi i}\int_{|\lambda|=\rho} \frac{S(\lambda)}{\lambda - z} d\lambda \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{S(\lambda)\lambda}{\lambda - z} dt \qquad (\lambda = \rho e^{it}) \\ &= \frac{1}{4\pi}\int_0^{2\pi} S(\lambda)\left[1 + \frac{\lambda + z}{\lambda - z}\right] dt \\ &= \frac{1}{4\pi i}\int_{|\lambda|=\rho} \frac{S(\lambda)}{\lambda} d\lambda + \frac{1}{4\pi}\int_0^{2\pi} S(\lambda)\frac{\lambda + z}{\lambda - z} dt \end{aligned}$$

$$= \frac{1}{2}R(0) + \frac{1}{4\pi}\int_0^{2\pi} S(\lambda)\frac{\lambda + z}{\lambda - z} dt, \tag{3}$$

where the curvilinear integrations are taken in the counterclockwise direction.

Suppose now that all the accumulation points of $\{\lambda_\nu\}$ form an uncountable set. Then the second principal part $\Psi(\lambda)$ of the given function $S(\lambda)$ never vanishes, as we already pointed out in the earlier discussion. If we put $z = re^{i\theta}$ for the above z, the point $\frac{\lambda\bar{\lambda}}{\bar{z}} = \frac{\rho^2}{r}e^{i\theta}$ lies outside the circle $|\lambda| = \rho$. Hence, as can be found immediately from Lemma [3] proved in the earlier discussion,

$$\begin{aligned} -\left\{\Phi\left(\frac{\lambda\bar{\lambda}}{\bar{z}}\right) + \Psi\left(\frac{\lambda\bar{\lambda}}{\bar{z}}\right)\right\} &= \frac{1}{2\pi i}\int_{|\lambda|=\rho} S(\lambda)\left(\lambda - \frac{\lambda\bar{\lambda}}{\bar{z}}\right)^{-1} d\lambda \\ &= \frac{1}{2\pi}\int_0^{2\pi} S(\lambda)\frac{\bar{z}}{\bar{z} - \bar{\lambda}} dt \qquad (\lambda = \rho e^{it}) \\ &= \frac{1}{4\pi}\int_0^{2\pi} S(\lambda)\left[1 - \frac{\bar{\lambda} + \bar{z}}{\bar{\lambda} - \bar{z}}\right] dt \\ &= \frac{1}{4\pi i}\int_{|\lambda|=\rho} \frac{S(\lambda)}{\lambda} d\lambda - \frac{1}{4\pi}\int_0^{2\pi} S(\lambda)\frac{\bar{\lambda} + \bar{z}}{\bar{\lambda} - \bar{z}} dt \\ &= \frac{1}{2}R(0) - \frac{1}{4\pi}\int_0^{2\pi} S(\lambda)\frac{\bar{\lambda} + \bar{z}}{\bar{\lambda} - \bar{z}} dt, \end{aligned}$$

so that

$$\Phi\left(\frac{\lambda\bar{\lambda}}{\bar{z}}\right) + \Psi\left(\frac{\lambda\bar{\lambda}}{\bar{z}}\right) + \frac{1}{2}R(0) = \frac{1}{4\pi}\int_0^{2\pi} S(\lambda)\frac{\bar{\lambda} + \bar{z}}{\bar{\lambda} - \bar{z}} dt. \tag{4}$$

Adding the equalities (3) and (4) term by term, we have

$$\Phi\left(\frac{\lambda\bar{\lambda}}{\bar{z}}\right) + \Psi\left(\frac{\lambda\bar{\lambda}}{\bar{z}}\right) + R(z) = \frac{1}{2\pi}\int_0^{2\pi} S(\lambda)\,\Re\left[\frac{\lambda + z}{\lambda - z}\right] dt. \tag{5}$$

Remembering that

$$\Phi(\lambda)=\sum_{\alpha=1}^{m}\sum_{\nu}\frac{c_{\alpha}^{(\nu)}}{(\lambda-\lambda_{\nu})^{\alpha}},$$

we obtain therefore

$$\Psi\left(\frac{\rho^{2}}{r}e^{i\theta}\right)=\frac{1}{2\pi}\int_{0}^{2\pi}S(\rho e^{it})\frac{\rho^{2}-r^{2}}{\rho^{2}+r^{2}-2\rho r\cos(\theta-t)}dt$$
$$-\frac{1}{2\pi}\int_{0}^{2\pi}S(\rho e^{it})\frac{\rho e^{it}}{\rho e^{it}-re^{i\theta}}dt-\sum_{\alpha=1}^{m}\sum_{\nu}c_{\alpha}^{(\nu)}\left(\frac{\rho^{2}}{r}e^{i\theta}-\lambda_{\nu}\right)^{-\alpha},$$

which shows that the desired equality (1) holds for every κ with $0<\kappa<1$ and every ρ with $\sup_{\nu}|\lambda_{\nu}|<\rho<\infty$.

Suppose next that all the accumulation points of $\{\lambda_{\nu}\}$ form a countable set. Then, as pointed out in the earlier discussion, $\Psi(\lambda)$ vanishes, and hence the desired equality (2) is deduced immediately from (1).

Corollary 1. If, in Theorem 4, there exist a positive number σ with $\sup_{\nu}|\lambda_{\nu}|<\sigma<\infty$ and countably infinite points $r_{j}e^{i\theta_{j}}$ with $\sup_{j}r_{j}<\sigma$ such that

$$\int_{0}^{2\pi}\frac{S(\sigma e^{it})}{\sigma e^{it}-r_{j}e^{i\theta_{j}}}dt=0\quad(j=1,2,3,\cdots),$$

then

$$(6)\qquad S\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{1}{2\pi}\int_{0}^{2\pi}S(\rho e^{it})\frac{1-\kappa^{2}}{1+\kappa^{2}-2\kappa\cos(\theta-t)}dt$$
$$(0<\kappa<1,\ \sup_{\nu}|\lambda_{\nu}|<\rho<\infty),$$

where the complex Poisson integral of S on the right-hand side converges uniformly to $R(0)$ or to $S(\rho e^{i\theta})$ according as κ tends to zero or to unity.

Proof. By hypothesis, it is a matter of simple manipulations to show that

$$\frac{1}{2\pi i}\int_{|\lambda|=\sigma}\frac{S(\lambda)}{\lambda-z_{j}}d\lambda=\frac{1}{2\pi i}\int_{|\lambda|=\sigma}\frac{S(\lambda)}{\lambda}d\lambda\quad(z_{j}=r_{j}e^{i\theta_{j}},\ j=1,2,3).$$

Accordingly we have $R(z_{j})=R(0)$, $j=1,2,3,\cdots$. In addition to it, $R(z)$ is regular on the domain $\{z:|z|<\infty\}$. As a result, $R(z)$ is a constant on the entire complex plane. Since, moreover, the equality (5) is rewritten in the form

$$S\left(\frac{\lambda\bar{\lambda}}{\bar{z}}\right)-R\left(\frac{\lambda\bar{\lambda}}{\bar{z}}\right)+R(z)=\frac{1}{2\pi}\int_{0}^{2\pi}S(\lambda)\Re\left[\frac{\lambda+z}{\lambda-z}\right]dt$$
$$(\lambda=\rho e^{it},\ z=re^{i\theta},\ r<\rho,\ \sup_{\nu}|\lambda_{\nu}|<\rho<\infty),$$

the desired equality (6) holds surely for every positive κ less than unity.

Next, from the relations

$$\frac{1}{2\pi}\int_0^{2\pi} S(\rho e^{it})dt=\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{S(\lambda)}{\lambda}d\lambda=R(0)$$

and the boundedness of $S(\lambda)$ on the circle $|\lambda|=\rho$, it is at once obvious that the complex Poisson integral on the right-hand side of (6) converges uniformly to $R(0)$ as κ tends to zero.

Since, furthermore, $S(\lambda)$ is regular on the circle $|\lambda|=\rho$ with $\sup_\nu |\lambda_\nu|<\rho<\infty$, the real and imaginary parts of $S(\rho e^{it})$ both are continuous as well as bounded on it and hence the Poisson integrals

$$\frac{1}{2\pi}\int_0^{2\pi}\Re[S(\rho e^{it})]\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\Im[S(\rho e^{it})]\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt$$

converge uniformly to $\Re[S(\rho e^{i\theta})]$ and $\Im[S(\rho e^{i\theta})]$ respectively as κ tends to unity, as can be easily verified with the aid of very small modifications of H. A. Schwarz's theorem [1] for Poisson's integral. This result shows that the complex Poisson integral on the right-hand side of (6) converges uniformly to $S(\rho e^{i\theta})$ as κ tends to unity.

The corollary has thus been proved.

Remark. This corollary is valid, of course, for the case where $\Psi(\lambda)$ vanishes.

Corollary 2. Let the hypothesis of Corollary 1 be satisfied, and let $M_S(\rho, 0)$ denote the maximum of the modulus $|S(\lambda)|$ of $S(\lambda)$ on the circle $|\lambda|=\rho$ with $\sup_\nu|\lambda_\nu|<\rho<\infty$. Then $M_S(\rho', 0)<M_S(\rho, 0)$ for any ρ' greater than ρ.

Proof. Since $S(\lambda)$ is regular on any closed annular domain $\{\lambda:\rho\leqq|\lambda|\leqq\rho'\}$ with $\sup_\nu|\lambda_\nu|<\rho<\rho'<\infty$ and since

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt=1,$$

the present corollary is an obvious consequence of the maximum-modulus principle and Corollary 1.

Theorem 5. Let $\{\lambda_\nu\}$, $S(\lambda)$, and $R(\lambda)$ be the same notations as those in Theorem 1 respectively; let $c(\neq\infty)$ be an arbitrarily given point on the complex plane; let C_1 be any positively oriented circle with center c and radius ρ_1 such that it contains $\{\lambda_\nu\}$ and all the accumulation points of $\{\lambda_\nu\}$ inside itself; let C_2 be any positively oriented circle with center c and radius ρ_2 less than ρ_1; let $M_S(\rho_1, c)=\max_{\lambda\in C_1}|S(\lambda)|$; and let $M_R(\rho_2, c)=\max_{\lambda\in C_2}|R(\lambda)|$. Then

$$M_R(\rho_2, c)<\frac{\rho_1 M_S(\rho_1, c)}{\rho_1-\rho_2}.$$

Proof. Since $R(\lambda)$ is regular on the domain $\{\lambda:|\lambda-c|<\infty\}$, we have

$$R(z)=\sum_{n=0}^{\infty}\frac{R^{(n)}(c)}{n!}(z-c)^n \quad (z\in C_2),$$

where

$$\frac{R^{(n)}(c)}{n!}=\frac{1}{2\pi i}\int_{C_1}\frac{S(\lambda)}{(\lambda-c)^{n+1}}d\lambda.$$

Since, moreover, the last equality yields

$$\frac{|R^{(n)}(c)|}{n!}=\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{S(\rho_1 e^{it}+c)}{\rho_1^n e^{int}}dt\right|\leqq\frac{M_S(\rho_1, c)}{\rho_1^n},$$

where the equality sign in the last relation applies if and only if the function $S(\rho_1 e^{it}+c)/\rho_1^n e^{int}$ is a constant on the closed interval $[0, 2\pi]$ of t [2], it is easily verified by direct computation that

$$|R(z)|<\frac{\rho_1 M_S(\rho_1, c)}{\rho_1-\rho_2}$$

for every $z\in C_2$. In consequence, we obtain the inequality required in the statement of the present theorem, as we wished to prove.

References

[1] C. Carathéodory: Theory of functions of a complex variable Vol. I, New York, 146–149 (1954).
[2] ——: Ibid., 114–116, 132.
[3] S. Inoue: Some applications of the functional-representations of normal operators in Hilbert spaces, Proc. Japan Acad., **38**(6), 263–268 (1962).