96. On Spherical Functions over $\mathfrak{p}$-adic Fields

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A $\mathfrak{p}$-adic analogue of the spherical function was first considered by Mautner[1] in the case of PL_2 and then by Tamagawa[2] for GL_n over any $\mathfrak{p}$-adic division algebra. The purpose of the present note is to show that the main part of the theory holds more generally under certain conditions, which are satisfied by almost all classical simple groups. It happened that similar considerations were contained in an independent work of Bruhat.[3]

1. *Notations and Assumptions.* Let k be a $\mathfrak{p}$-adic number field. We denote by $\mathfrak{o}$ and $\mathfrak{p}=(\pi)$ the valuation ring in k and its unique prime ideal, respectively, π denoting a prime element. Let G be an algebraic subgroup of $GL_n(k)$, defined over k, and set as follows:

A= the connected component of the identity (in the sense of Zariski topology) of the subgroup of G consisting of all diagonal matrices in G,

N= the subgroup of G consisting of all upper unipotent matrices (i.e. matrices $x=(x_{ij})$ such that $x_{ii}=1$, $x_{ij}=0$ $(i>j)$) in G,

U= the subgroup of G consisting of all $\mathfrak{o}$-unit matrices (i.e. matrices $x=(x_{ij})$ such that $x_{ij}\in\mathfrak{o}$ and $\det x\notin\mathfrak{p}$) in G.

In the following, we assume that G satisfies the following two conditions (I), (II).

$$\text{(I)}\qquad G=U\cdot AN=U\cdot A\cdot U.$$

This implies that U is a maximal (open) compact subgroup of G (viewed as a topological group in the $\mathfrak{p}$-adic topology), and that A, N are, respectively, a maximal trivial torus and a maximal unipotent subgroup of G (viewed as a linear algebraic group over k). Moreover, it follows from the existence of maximal compact subgroup that G is reductive and hence unimodular. We normalize the (both side invariant) Haar measure dx on G in such a way that $\int_U dx=1$.

Now, for $(m)=(m_1,\cdots,m_n)\in \boldsymbol{Z}^n$ ($\boldsymbol{Z}$=ring of rational integers), put $\pi^{(m)}=\mathrm{diag.}(\pi^{m_1},\cdots,\pi^{m_n})$ and call A_π the subgroup of A consisting of all matrices of this form in A. If $\dim A=\nu$, we have $A\cong(k^*)^\nu$, k^* denoting the multiplicative group of non-zero elements in k, so that there exists a lattice M of rank ν in $\boldsymbol{Z}^n$ such that $A_\pi=\{\pi^{(m)}\mid(m)\in M\}$. For $u\in U$, normalizing A, we have

$$u\pi^{(m)}u^{-1}=\pi^{w(m)}\quad\text{for}\quad(m)\in M,$$

with an automorphism w of M. The w's obtained in this way form a finite group W of automorphisms of M, called the 'Weyl group'. Put further

$$\Lambda=\{(m)\in M \mid w(m)\leq(m) \quad \text{for all } w\in W\},$$

$\leq$ denoting the lexicographical linear order in $\boldsymbol{Z}^n$. Then, our second assumption may be stated as follows:

(II) For $(m)\in\Lambda$, we have $m_1\geq m_2\geq\cdots\geq m_n$.

This implies, in particular, that

$$G=\bigcup_{(m)\in\Lambda} U\pi^{(m)}U \qquad \text{(disjoint union)}.$$

2. *Results*. Let $L(G, U)$ denote the ring of all complex-valued continuous functions f on G, with compact carriers, satisfying the relation

$$(1) \qquad f(uxu')=f(x) \quad \text{for all } x\in G,\ u, u'\in U,$$

the product in $L(G, U)$ being defined by the convolution on G. Then we have

THEOREM 1. *$L(G, U)$ is an affine ring of dimension ν over $\boldsymbol{C}$* (=field of complex numbers) (i.e. a commutative integral domain with 1, which is finitely generated over $\boldsymbol{C}$ and of transcendence degree ν over $\boldsymbol{C}$).

A complex-valued continuous function ω on G is called a *zonal spherical function* (abbreviated as z.s.f.) on G relative to U, if it satisfies the condition (1) (f being replaced by ω) and if the mapping

$$L(G, U)\ni f\to\hat{\omega}(f)=\int_G f(x)\omega(x^{-1})dx$$

is a homomorphism (of algebra) from $L(G, U)$ onto $\boldsymbol{C}$.[4] Since U is open in G, it can be proved conversely that any homomorphism $\hat{\omega}$: $L(G, U)\to\boldsymbol{C}$ is obtained in this way from a uniquely determined z.s.f. ω on G relative to U. Therefore, the set $\Omega(G, U)$ of all z.s.f. on G relative to U may be regarded as a model of the affine ring $L(G, U)$. Then we have

THEOREM 2. *$\Omega(G, U)$ is analytically isomorphic to a quotient space of the form $W\diagdown(\boldsymbol{C}^*)^\nu$.*

3. *Outline of the Proof*. First we construct z.s.f. depending on ν-complex parameters as follows. Let α be a homomorphism from A_π into $\boldsymbol{C}^*$. Writing $x\in G$ in the form $x=u\pi^{(m)}n$ with $u\in U$, $n\in N$, put $\psi_\alpha(x)=\alpha(\pi^{(m)})$. Then it is easy to see that

$$(2) \qquad \omega_\alpha(x)=\int_U \psi_\alpha(x^{-1}y)dy$$

becomes a z.s.f. on G relative to U. Now, take a basis $\{(m^{(i)})\}_{1\leq i\leq\nu}$ of M, call δ a continuous homomorphism from A into $\boldsymbol{R}^+$(=multiplicative group of positive real numbers) defined by $d(ana^{-1})=\delta(a)dn$, dn being a Haar measure on N, and let q be any fixed positive number. We assign to α a ν-tuple $\boldsymbol{s}=(s_1,\cdots,s_\nu)\in \boldsymbol{C}^\nu$ by the relation

$$\delta^{\frac{1}{2}}\alpha(\pi^{(m^{(i)})})=q^{s_i},$$

s_i being determined modulo $(2\pi\sqrt{-1}/\log q)\cdot\boldsymbol{Z}$. This gives a 1-1 correspondence between $\mathrm{Hom}(A_\pi, \boldsymbol{C}^*)$ and $\boldsymbol{C}^\nu(2\pi\sqrt{-1}/\log q)\cdot\boldsymbol{Z}^\nu\cong(\boldsymbol{C}^*)^\nu$. Hence we write $\omega_{\boldsymbol{s}}$, instead of ω_α, where α is corresponding to $\boldsymbol{s}\in\boldsymbol{C}^\nu$ in the above sense. Moreover, by means of the natural pairing $M\times\boldsymbol{C}^\nu\ni((m),\boldsymbol{s})\to(m)\cdot\boldsymbol{s}\in\boldsymbol{C}$ defined by $(m^{(i)})\cdot\boldsymbol{s}=s_i$ $(1\leq i\leq\nu)$, W is made to operate on $\boldsymbol{C}^\nu$ as a group of linear transformations.

In these notations, it can first be proved that $\omega_{w\boldsymbol{s}}=\omega_{\boldsymbol{s}}$, in a similar way as in the real case.[5] It follows that, for $f\in L(G,U)$, the Fourier transform $\hat{f}(\boldsymbol{s})=\hat{\omega}_{\boldsymbol{s}}(f)$ belongs to the ring $\boldsymbol{C}[q^{\pm s_1},\cdots,q^{\pm s_\nu}]^W$ of all W-invariant Fourier polynomials in $q^{\pm s_i}$. In particular, for the characteristic function $c_{(r)}$ of $U\pi^{(r)}U((r)\in\Lambda)$, we have

$$\hat{c}_{(r)}(\boldsymbol{s})=\sum_{(m)\in\Lambda}\gamma_{(r)(m)}\sum_{w\in W/W_{(m)}}q^{w(m)\cdot\boldsymbol{s}},$$

$$\gamma_{(r)(m)}=\delta^{-\frac{1}{2}}(\pi^{(m)})\cdot\int_{U\pi^{(m)}N\cap U\pi^{(r)}}dx,$$

$W_{(m)}$ denoting the subgroup of W formed of all the w leaving (m) fixed. Now it can easily be seen that the infinite matrix $(\gamma_{(r)(m)})$ with the indices (r), $(m)\in\Lambda$ ordered in the lexicographic order $\leq$ is of lower triangular form and non-singular. From this we conclude that *the mapping* $f\to\hat{f}$ *is an isomorphism* (with respect to the structure of algebra over $\boldsymbol{C}$) *from* $L(G,U)$ *onto* $\boldsymbol{C}[q^{\pm s_1},\cdots,q^{\pm s_\nu}]^W$. Theorem 1 follows then immediately. It implies also that any z.s.f. can be written in the form $\omega_{\boldsymbol{s}}(\boldsymbol{s}\in\boldsymbol{C}^\nu)$; whence follows Theorem 2.

4. *Remarks.* The conditions (I), (II) are satisfied, for instance, by $GL_n(k)$, $SL_n(k)$, $Sp_n(k)$, $SO_n(k,F)$, taken in a suitable matricial expression. For other types of classical groups, we have to modify these conditions, in replacing A by a certain reductive algebraic subgroup H without unipotent element, of G, containing A (for most cases, by $Z(A)$); then all the above results remain true. In each case, it is not hard to determine the structure of $L(G,U)$ explicitly. However, to determine the Plancherel measure and the explicit form of the z.s.f., it seems indispensable to know $\gamma_{(r)(m)}$ more closely, and this is still an open problem for $\nu>1$. Our theory has some applications to the theory of Hecke operator, which, together with a full account of the above results, will be published elsewhere.

References

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