148. *Configuration Theorems in Affine Plane*

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The axiomatic theory of Euclidean geometry originated by D. Hilbert has led a series of configurational propositions. In 1943, M. Hall [3] has given a new method of coordinating a projective plane. By Hall method, many configurational propositions have been obtained by L. A. Skorniakov [4], B. I. Argunov [1]. In his interesting article [2], R. H. Bruck has given the foundation of an affine plane by Hall method. In this paper, we shall consider configurational propositions in an affine plane.

For the completeness, we shall give some definitions briefly. An *affine plane* π is a set of points and lines among which a relation on incidence satisfies the following conditions:

1. *Two distinct points are incident with one and only one line.*
2. *Two distinct lines are incident with at most one point.*
3. *Through a point exterior to a given line there is exactly one line having no intersection point with the given line.*
4. *There exist four points in general position* (no three of four points are incident with one line).

If there is no point on two lines a and b, then we call a and b parallel and write $a//b$. As well known, $a//b$ and $b//c$ implies $a//c$. Every line has same number of points, every point lies on same number of lines.

Let X, Y, O, E be four points in general position on π. We call the point O the origin of coordinates, line OE the unit line. Let $\boldsymbol{M}$ be the set of symbols which is 1–1 correspondence with the point set of the line OE. Let 0(1) by the element of $\boldsymbol{M}$ with corresponds with $O(E)$. We denote by $a, b, c, \cdots$ the element of $\boldsymbol{M}$. By the well-known method (see R. H. Bruck [2]), we can define a ternary ring R on π. We may introduce coordinates in an affine plane π in the following way (see Figs. 1, 2).

We take three distinct lines OA, OB, and OE through O which we call the *x-axis*, the *y-axis* and the *unit line*. To each point A of OE, we assign a coordinate $A(a, a)$, $a \in \boldsymbol{M}$. For any point P, we define its coordinate $P(a, b)$ as Fig. 1, where $OY//AP$, $OX//BP$. If the line parallel to the y-axis is incident with a given point $P(a, b)$, the equation of the line is given by $x=a$. Suppose that a line l intersects the y-axis, and let l' be through the origin parallel to the line

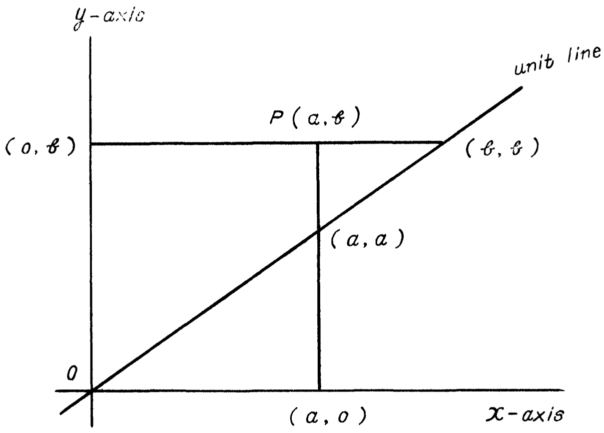

Fig. 1

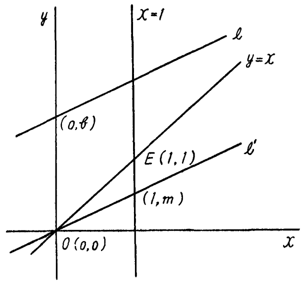

Fig. 2

l. The *slope* of a line l intersecting of y-axis is defined as the element $m \in \boldsymbol{M}$, where $(1, m)$ is the coordinate of the intersection point of the line $x=1$ and the line l'. Two lines are parallel, if and only if, their slopes are equal. If the given line is parallel to the y-axis, the slope is meaningless. The slope of the line parallel to x-axis is 0. Any line (not parallel to y-axis) l is uniquely defined by the slope and the intersection point $(0, b)$ of l and y-axis. Then *the equation of the line* l is given by a ternary operations:

$$y = T(x, m, b).$$

Then, the ternary operation defined on an affine plane π satisfies the fundamental properties of *ternary ring* (see M. Hall [3], L. A. Skorniakov [4]). For a ternary ring, the addition $a+b$ of a and b is defined by

$$a+b = T(a, 1, b),$$

the multiplication ab by

$$ab = T(a, b, 0).$$

For a ternary ring on an affine plane π,

$$T(a, b, c) = ab + c \quad \text{and} \quad (a+b)c = ab + ac$$

hold universally, if and only if the Desargues affine small theorem is provable (see R. H. Bruck [2]). We can prove that for the ternary ring on π, $ab+ac=a(b+c)$, or $T(a, b, ac) = aT(1, b, c)$ is provable, if and only if the Desargues small theorem with the perspective center holds on π (see R. H. Bruck [2]).

If the Desargues small theorem with perspective center (see Fig. 3) holds on π, then we can prove the associative law for multiplication. To prove that the multiplication is associative, we consider $y=x$, $y=xm$, $y=xn$ as the Fig. 4. Let $A(1, m)$ be the point of the intersection of $x=1$ and $y=xm$. Draw the line l through the point $B(m, m)$ parallel to the y-axis. Let D be the intersection of the line

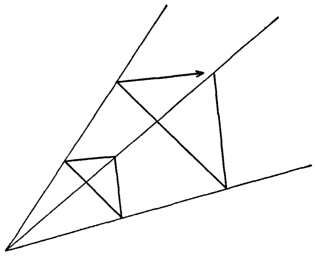

Fig. 3

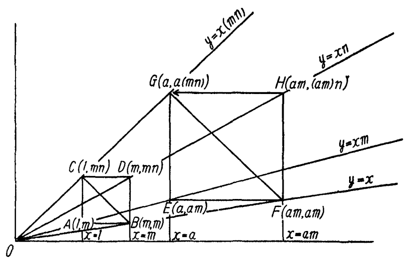

Fig. 4

l and $y=xn$, then we have $D(m, mn)$. Passing through the point D, we make the line $y=mn$, and let us take $C(1, mn)$. Then the line joining O and C is $y=x(mn)$. Take any element a of $\boldsymbol{M}$, and let the points E, G be the intersection points of $x=a$ and $y=xm$, $x=a$ and $y=x(mn)$ respectively, then we have $E(a, am)$, $G(a, a(mn))$. Let F be the intersection point of $y=am$ and $y=x$, then $F(am, am)$. For the point H of intersection of $x=am$ and $y=xn$, $H(am, (am)n)$ (see Fig. 4). Then two triangles ABC and EFG have the perspective center O, and $AB//EF$, $AC//EG$. Hence we have $BC//FG$. Next we consider two triangles BCD, FGH, then O is the perspective center of these triangles, and $BD//FH$, $BC//FG$. Therefore we have $CD//GH$. Hence GH is parallel to the x-axis. This show that y-coordinates of G and H are same. Therefore we have $a(mn)=(am)n$. We have the following

Theorem. A ternary ring $\boldsymbol{M}$ *on an affine plane* π *is the division algebra, if and only if the Desargues small theorem with perspective center holds.*

References

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