144. On Cohomological Dimension for Paracompact Spaces. II

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The present note is a continuation of our previous paper on the cohomological dimension of paracompact spaces [12]. In the previous one we proved the following theorem:

Theorem 1.[1] (*Sum*) *Let X be a space and let $\{X_k\}$ be a countable collection of closed sets of X such as $\bigcup_{k=1}^{\infty} X_k = X$. If each X_k has $D(X_k; G) \leqq n$, then we have $D(X; G) \leqq n$.*

Hence, we shall prove the sum theorems of the another forms.

All spaces will be assumed to be paracompact Hausdorff spaces and all coefficient groups will be assumed to be non-zero additive Abelian groups.

Let $D(X; G)$[2] be the cohomological dimension with coefficient group G defined as follows: $D(X; G) \leqq n$ if and only if for any closed set A of X and for any integer m such as $m \geqq n$ the homomorphism $H^m(X; G) \rightarrow H^m(A; G)$ induced by inclusion is onto where n is a non-negative integer and $H^m(X; G)$, $H^m(A; G)$ are n-th Čech cohomology groups with coefficient group G.

We state now for reference the following two theorems to be used below.

Theorem 2.[3] (*Mayer-Vietoris*) *If X is a space and if X_1, X_2 are closed sets of X such as $X = X_1 \smile X_2$, then the following sequence is exact*

$$\cdots \rightarrow H^n(X; G) \rightarrow H^n(X_1; G) \times H^n(X_2; G) \rightarrow H^n(X_1 \frown X_2; G) \rightarrow H^{n+1}(X; G) \rightarrow \cdots.$$

Theorem 3.[4] (*Katetov*) *If X is a collectionwise normal Hausdorff space, then for each closed set S of X and for each locally finite open covering $\{U_\xi\}$ of S there exists a locally finite collection $\{V_\xi\}$ of open sets of X satisfying the following condition*

$$C(S, \{U_\xi\}, \{V_\xi\}): \bigcup_\xi V_\xi \supset S,\ V_\xi \frown S \subset U_\xi \text{ and the correspondence } V_\xi \leftrightarrow U_\xi \text{ induces } \{V_\xi\} \mid S \cong \{U_\xi\} \cong \{V_\xi\} \cong \{\overline{V}_\xi\}$$

where $\{U_\xi\} \cong \{V_\xi\}$ denotes that $\{U_\xi\}$ is similar to $\{V_\xi\}$.

1) Cf. [12, Theorem 3.2].

2) Cf. [12, Definition]. The cohomological dimension for compact spaces can be seen in [1] and [6].

3) Cf. [3] and [5, p. 43].

4) Cf. [7, Theorem 3.2].

All notations which will be used below are the notations used in [12].

Let F be a closed set of a space X. Let us define $D(X, F; G)$ as follows: $D(X, F; G) \leqq n$ if and only if $D(C; G) \leqq n$ for every closed set C of X such as $C \subset X-F$.[5]

Theorem 4. *Let X be a space and let F be a closed set of X. Then we have $D(X; G) = max\{D(F; G),\ D(X, F; G)\}$.*[6]

Proof. Let $\max\{D(F; G), D(X, F; G)\} = n$ and let A be an arbitrary closed set of X. It is enough to show that an arbitrary element e of $H^m(A; G)$ $(m \geqq n)$ can be extended to an element of $H^m(X; G)$. Note the following exact sequence by Theorem 2.

$$\cdots \to H^m(A \smile F; G) \to H^m(A; G) \times H^m(F; G) \to H^m(A \frown F; G) \to \cdots.$$

Let $e_1 = e \mid A \frown F$.[7] Since $D(F; G) \leqq n$, e_1 can be extended to e_2 of $H^m(F; G)$. Now by the exactness of the above sequence there is an e_3 in $H^m(A \smile F; G)$ such as $e_3 \mid A = e$. Let α be a locally finite open covering of $A \smile F$ such that z_α^m is an m-cocycle of $N(\alpha)$[8] and $\{z_\alpha^m\}$[9] $= e$. By Theorem 3 there exists a locally finite collection β of open sets of X satisfying the condition $C(F \smile A, \alpha, \beta)$. By the normality of X there is an open set H such that $F \smile A \subset H \subset \overline{H} \subset B$ where $B = \smile\{U \mid U \in \beta\}$. Note the following exact sequence by Theorem 2,

$$\cdots \to H^m(X; G) \to H^m(\overline{H}; G) \times H^m(X-H; G) \to H^m(\overline{H}-H; G) \to \cdots.$$

Let $e_4 = \{i(A \smile F, \overline{H}) z_\alpha^m\} \mid \overline{H} - H$.[10] Since $X - H \subset X - F$ and hence $D(X-H; G) \leqq n$, e_4 can be extended to e_5 of $H^m(X-H; G)$. By the exactness there is an e_6 in $H^m(X; G)$ such as $e_6 \mid H = \{i(A \smile F, \overline{H}) \pi_{\alpha, \beta \mid A \smile F} z_\alpha^m\}$.[11] Hence $e_6 \mid A = \{i(A \smile F, \overline{H}) z_\alpha^m\} \mid A = e$. This means that $D(X; G) \leqq \max\{D(F; G),\ D(X, F; G)\}$ completing the proof.

Using Theorem 1 and the above theorem, we have the following corollary.

Corollary. *If $\{X_n\}$ is a countable closed covering of X such that $X_k \subset X_{k+1}$ $(k = 1, 2, \cdots)$, then $D(X; G) = max\{D(X_{k+1}, X_k; G)$ where we put $X_0 = \phi$.*[12]

Let α be a point finite open covering of X and let x be an arbitrary point of X. Let us denote by $ord(x : \alpha)$ the integer n such

5) We can see the definition of this form for dim in [10].

6) The theorem of this form for dim was proved in [10, Lemma 4].

7) $e \mid A \frown F$ denotes the image of e by the homomorphism $H^m(A; G) \to H^m(A \frown F; G)$ induced by inclusion (cf. [12]).

8) $N(\alpha)$ is the nerve of α.

9) $\{z_\alpha^m\}$ denotes the element of $H^m(A \smile F; G)$ containing Z_α^m (cf. [12]).

10) $i(A \smile F, \overline{H})$ is the homomorphism of the cocycles of $N(\alpha)$ into cocycles of $N(\beta)$ induced by the correspondence $\alpha \cong \beta$ in $C(A \smile F, \alpha, \beta)$ (cf. [12]).

11) $\beta \mid A \smile F = \{U \frown (A \smile F) \mid U \in \beta\}$. $\pi_{\alpha, \beta \mid A \smile F}$ denotes the homomorphism of the cocycles of $N(\alpha)$ into the cocycles of $N(\beta \mid A \smile F)$ induced by inclusion (cf. [12]).

12) The theorem of this form for dim was proved in [10, Lemma 4].

that x is contained in at most n distinct elements of α.

Theorem 5. *Let X be a space and let $\alpha=\{U_\lambda \mid \lambda \in \Lambda\}$ be a locally finite open covering of X such that for each $\lambda \in \Lambda$ $D(\bar{U}_\lambda; G) \leqq n$. Then we have $D(X; G) \leqq n$.*[13]

Proof. For each natural number k we denote by Φ_k the collection of all subsets of Λ each of which are distinct k elements of Λ.

If we put $T_1=\{x \mid ord(x:\alpha)=1\}$, then T_1 is a closed set of X. Let $\mathfrak{F}_1=\{F^1_\varphi \mid \varphi \in \Phi_1\}$ where $F^1_\varphi = T_1 \frown U_\lambda$ such as $\varphi=\lambda$. Then $\mathfrak{F}_1$ is a discrete collection[14] of closed sets of X. Since X is collectionwise normal, there exists a collection $\beta_1=\{V^1_\varphi \mid \varphi \in \Phi_1\}$ of open sets of X such that $F^1_\varphi \subset V^1_\varphi \subset$ some U_λ for each $\varphi \in \Phi_1$, and the collection $\bar{\beta}_1 = \{\bar{V}^1_\varphi \mid \varphi \in \Phi_1\}$ is discrete. From $D(\bar{V}^1_\varphi; G) \leqq$[15] $D(\bar{U}_\lambda; G) \leqq n$ we have $D(\bigcup_{\varphi \in \Phi_1} \bar{V}^1_\varphi; G) \leqq n$.[16] Let $V^1 = \bigcup_{\varphi \in \Phi_1} V^1_\varphi$. Then we have $V^1 \supset T_1$.

Now let us suppose that for each $k=1, 2, \cdots, l-1$ we have constructed T_k, $\mathfrak{F}_k$, β_k and V^k such that

$(1)_k$: T^k is a closed set of X,

$(2)_k$: $\beta_k=\{V^k_\varphi \mid \varphi \in \Phi_k\}$ is a collection of open sets of X such that $\bar{\beta}_k$ is discrete, and $\bigcup_{k=1}^{k} V^k \supset T_k$,

and

$(3)_k$: V^k is an open set of X such as $D(\bar{V}^k : G) \leqq n$.

Let $T_l=\{x \mid ord(x:\alpha) \leqq l\}$. Then T_l is closed in X. Because, for any point x of $X-T_l$ there exists an element φ of Φ_{l+1} and for this φ the neighborhood $\bigcap_{\lambda \in \varphi} U_\lambda$ of x is disjoint from T_l. So we have T_l satisfying $(1)_l$.

Next, we shall construct β_l and prove that β_l satisfies $(2)_l$. Let $\mathfrak{F}_l=\{F^l_\varphi \mid \varphi \in \Phi_l\}$ where $F^l_\varphi = T_l \frown (\bigcup_{\lambda \in \varphi} U_\lambda) - \bigcup_{h=1}^{l-1} V^h$ for each $\varphi \in \Phi_l$. Then $\mathfrak{F}_l$ is a discrete collection of closed sets of X. To prove this fact we divide three parts: (i) $\mathfrak{F}_l$ is locally finite, (ii) $\mathfrak{F}_l$ is a disjoint collection, and (iii) F^l_φ is closed for each $\varphi \in \Phi_l$. If we note that $F^l_\varphi \subset \bigcap_{\lambda \in \varphi} U_\lambda$ and $\{\bigcap_{\lambda \in \varphi} U_\lambda \mid \varphi \in \Phi_l\}$ is locally finite in X, then we immediately have (i). For any distinct two element φ, φ' of Φ_l there is a λ_0 such as $\lambda_0 \in \varphi$, $\lambda_0 \notin \varphi'$ and from the construction of T_l we have $F^l_\varphi \frown F^l_{\varphi'} \subset T_l \frown (\bigcap_{\lambda \in \varphi'} U_\lambda) \frown U_{\lambda_0} = \phi$. Therefore, we have (ii). Let x be an arbitrary point of $X - F^l_\varphi$ $(\varphi \in \Phi_l)$. If x is contained in $\bigcup_{h=1}^{l-1} V^h$, then $\bigcup_{h=1}^{l-1} V^h$ is a

13) The theorem of this form with respect to dim was proved in [9, Theorem 2] and we have the theorem with respect to Ind of totally normal space in [11, Theorem 5].

14) Discrete collection is the locally finite collection of mutually disjoint sets.

15) Cf. [12, Theorem 3.1].

16) Cf. [12, Corollary 3.3].

desirable neighborhood of x. Let us suppose that x is not contained in $\bigcup_{h=1}^{l-1} V^h$. If x is contained in T_l, then there exists a $\varphi' \in \Phi_l$ such as $x \in \bigcap_{\lambda \in \varphi'} U_\lambda$. From $x \notin F^l_\varphi$ we get $\varphi \neq \varphi'$ and, therefore, there exists a $\lambda_0 \in \Lambda$ such as $\lambda_0 \in \varphi'$, $\lambda_0 \notin \varphi$. Since $F^l_\varphi \frown U_{\lambda_0} = \phi$, U_{λ_0} is a desirable neighborhood of x. In the case $x \notin \bigcup_{h=1}^{l-1} V^h$ and $x \notin T_l$ there exist mutually distinct elements $\lambda_1, \cdots, \lambda_{l+1}$ of Λ such as $x \in \bigcap_{h=1}^{l+1} U_{\lambda h}$. Then $\bigcap_{h=1}^{l+1} U_{\lambda h}$ is a desired neighborhood of x. Here, we get (i), (ii), and (iii) for $\mathfrak{F}_l$. Since X is collectionwise normal, there exists a collection $\beta_l = \{V^l_\varphi \mid \varphi \in \Phi_l\}$ of open sets of X such that $F^l_\varphi \subset V^l_\varphi \subset \overline{V}^l_\varphi \subset$ some U_λ for each $\varphi \in \Phi_l$ and $\overline{\beta}_l$ is discrete in X. Let $V^l = \bigcup_{\varphi \in \Phi_l} V^l_\varphi$. Then by the assumption $T_{l-1} \subset \bigcup_{k=1}^{l-1} V^k$ we get $T_l \subset (T_l - T_{l-1}) \smile (\bigcup_{k=1}^{l-1} V^k) \subset V^l \smile (\bigcup_{k=1}^{l-1} V^k)$ and, hence we obtain $(2)_l$.

Finally, we shall show $(3)_l$. Since for each $\varphi \in \Phi_l$, $\overline{V}^l_\varphi \subset \overline{U}_\lambda$ for some $\lambda \in \Lambda$, we have $D(\overline{V}^l_\varphi; G) \leqq$[15] $D(\overline{U}_\lambda; G) \leqq n$. By $(2)_l$ we have $D(\overline{V}^l; G) \leqq$[16] $D(\bigcup_{\varphi \in \Phi_l} \overline{V}^l_\varphi; G) \leqq n$.

Since α is a locally finite open covering of X, we have $X = \bigcup_{k=1}^{\infty} T_k$ and, hence we have $X = \bigcup_{k=1}^{\infty} V^k$. By Theorem 1 we obtain $D(X; G) \leqq D(\bigcup_{k=1}^{\infty} \overline{V}^k; G) \leqq n$. This completes the proof.

Let X be a space. Now we define a local[17] cohomological dimension $loc\, D(X; G)$ as follows: Let $loc\, D(X; G) \leqq n$ $(n \geqq -1)$ if and only if for any point x of X there exists a neighborhood U_x of x such as $D(\overline{U}_x; G) \leqq n$.

Theorem 6. *Let X be a space. Then we have $D(X; G) = loc\, D(X; G)$.*[18]

Proof. We can easily see $loc\, D(X; G) \leqq D(X; G)$.[15] Conversely, if we assume $loc\, D(X; G) \leqq n$, for each point x of X there exists a neighborhood U_x of x such that $D(\overline{U}_x; G) \leqq n$. Since X is paracompact, we obtain an open covering of X satisfying the conditions of Theorem 5. Hence we have $D(X; G) \leqq loc\, D(X; G)$.

Using the above theorem we get the following theorem:

Theorem 7. *Let X be a space and let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a locally countable closed covering of X such that $D(F_\lambda; G) \leqq n$ for each $\lambda \in \Lambda$.*

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17) Local dimensions for dim and Ind were defined in [4].

18) We have the theorem of this form with respect to dim [3, [3.3]], and for paracompact totally normal space we have the theorem with respect to Ind in [3, [3.4]].

Then we have $D(X; G) \leqq n$.

Proof. By the assumption of $\{F_\lambda \mid \lambda \in \Lambda\}$ for any point of X there exists a neighborhood U_x of x such that $\overline{U}_x \frown F_\lambda \neq \phi$ for only countable $\lambda = \lambda_1, \lambda_2, \cdots$. By Theorem 1 we obtain $D(\overline{U}_x; G) = D(\bigcup_{k=1}^{\infty} (\overline{U}_x \frown F_{\lambda k}); G) \leqq n$ and hence we have $loc\, D(X; G) \leqq n$. By theorem 6 we have $D(X; G) = loc\, D(X; G) \leqq n$.

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