135. A Proof of Kotaké and Narasimhan's Theorem

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We shall give a simple proof of the following theorem announced by Kotaké and Narasimhan [1].

Theorem. *Let $P=P(x, D)$ be a linear elliptic differential operator of order m with analytic coefficients in a domain $\Omega \subset R^n$. Then, a function $u=u(x)$ is analytic in Ω if and only if it satisfies*

(1) $$\|P^p u\|_{L^2(G)} \leqq B^{p+1}(pm)! \quad (p=0, 1, 2, \cdots)$$

for every relatively compact subdomain $G \subset \Omega$ with a constant B depending only P, G and u.

Proof of Sufficiency. u is in $C^{pm-[n+1]/2}(\Omega)$ if $P^p u$ is in $L^2_{\mathrm{loc}}(\Omega)$. Therefore we may suppose that u is infinitely differentiable.

For functions f in $C^\infty(G)$ we define

$$\|\nabla^q f\|_\delta = \sum_{|\alpha|=q} \|D^\alpha f\|_{L^2(G_\delta)},$$

where G_δ is the set of points $x \in G$ such that the distance from x to the boundary of G is larger than δ. We shall make use of the following apriori inequalities (see [3] for a proof).

(2) $$\|\nabla^m f\|_{\delta+\sigma} \leqq C(\|Pf\|_\sigma + \delta^{-m}\|f\|_\sigma),$$

(3) $$\|\nabla^{m-r} f\|_{\delta+\sigma} \leqq C\varepsilon^r(\|\nabla^m f\|_\sigma + (\delta^{-m}+\varepsilon^{-m})\|f\|_\sigma) \quad (0 \leqq r \leqq m).$$

ε may take an arbitrary positive number and the constant C depends only on P and G.

We fix a positive constant ρ and define the semi-norm $N^{pm}(u)$ by

$$N^{pm}(u) = \sup_{\delta \leqq \rho} \delta^{pm} \|\nabla^{pm} u\|_\delta.$$

First we shall prove that if ρ is sufficiently small, then

(4) $$N^{pm}(u) \leqq C_0 \left\{ N^{(p-1)m}(Pu) + \sum_{q=0}^{p-1} \frac{(pm)!}{(qm)!} N^{qm}(u) \right\}$$

holds for every $u \in C^\infty(G)$ with a constant C_0 independent of u and $p=1, 2, \cdots$.

When $p=1$, (4) is obviously valid with $C_0=2^m C$. In case $p+1 \geqq 2$, it follows from (2) that

$$N^{(p+1)m}(u) = \sup_{(p+2)\delta \leqq \rho} ((p+2)\delta)^{(p+1)m} \|\nabla^{(p+1)m} u\|_{(p+2)\delta}$$
$$\leqq 9^m C \sup_{(p+2)\delta \leqq \rho} (p\delta)^{(p+1)m} \{\|P\nabla^{pm} u\|_{(p+1)\delta} + \delta^{-m} \|\nabla^{pm} u\|_{(p+1)\delta}\}.$$

Because of the analyticity of the coefficients of $P(x, D)$, their r-th derivatives are majorated by $A^{r+1} r!$ with a constant $A \geqq 1$.

Leibniz' formula gives

$$\|P\nabla^{pm} u\|_{(p+1)\delta} \leqq \|\nabla^{pm} Pu\|_{(p+1)\delta} + \sum_{r=1}^{pm} \binom{pm}{r} \|P^{[r]} \nabla^{pm-r} u\|_{(p+1)\delta}$$

$$\leqq \|\nabla^{pm}Pu\|_{(p+1)\delta}+\sum_{r=1}^{pm}\binom{pm}{r}A^{r+1}r!\sum_{s=0}^{m}\|\nabla^{pm+s-r}u\|_{(p+1)\delta}$$

$$\leqq \|\nabla^{pm}Pu\|_{(p+1)\delta}+(m+1)\sum_{q=1}^{p}\sum_{t=1}^{m}\frac{(pm)!}{(qm-t)!}A^{(p-q)m+t+1}\|\nabla^{(q+1)m-t}u\|_{(p+1)\delta}$$

$$+m\sum_{t=1}^{m}(pm)!\,A^{pm+1}\|\nabla^{m-t}u\|_{(p+1)\delta}.$$

Let $C_2=(m+1)9^mA^{m+1}C$. For $r=1,2,\cdots,m$, we have by (3)

$$C_2\frac{(pm)!}{(pm-r)!}(p\delta)^{(p+1)m}\|\nabla^{(p+1)m-r}u\|_{(p+1)\delta}$$

$$\leqq CC_2(pm)^r\varepsilon^r\{(p\delta)^{(p+1)m}\|\nabla^{(p+1)m}u\|_{p\delta}+(p^m+(p\delta)^m\varepsilon^{-m})(p\delta)^{pm}\|\nabla^{pm}u\|_{p\delta}\}.$$

Hence by substituting $(pm)^{-1}(2mCC_2)^{-1/r}$ for ε and summing over r, we have

$$\sum_{r=1}^{m}C_2\frac{(pm)!}{(pm-r)!}(p\delta)^{(p+1)m}\|\nabla^{(p+1)m-r}u\|_{(p+1)\delta}$$

$$\leqq\frac{1}{2}\{(p\delta)^{(p+1)m}\|\nabla^{(p+1)m}u\|_{p\delta}+(p^m+(pm\rho)^m(2mCC_2)^m)(p\delta)^{pm}\|\nabla^{pm}u\|_{p\delta}\}$$

$$\leqq\frac{1}{2}N^{(p+1)m}(u)+C_3\frac{((p+1)m)!}{(pm)!}N^{pm}(u),$$

where C_3 is a constant depending only on C, C_2, m and ρ.

For $q=p-1,p-2,\cdots,1$, we have

$$C_2\frac{(pm)!}{(qm-t)!}A^{(p-q)m}(p\delta)^{(p+1)m}\|\nabla^{(q+1)m-t}u\|_{(p+1)\delta}$$

$$\leqq C_2\frac{((p+1)m)!}{((q+1)m)!}(A\rho)^{(p-q)m}\left(\frac{p}{p+1}\,\frac{q+1}{q}\right)^{(q+1)m}$$

$$\times\frac{(qm)!}{(qm-t)!}(q\delta')^{(q+1)m}\|\nabla^{(q+1)m-t}u\|_{(q+1)\delta'}$$

$$\leqq C_4\frac{((p+1)m)!}{((q+1)m)!}N^{(q+1)m}(u)+C_4\frac{((p+1)m)!}{(qm)!}N^{qm}(u).$$

The constant C_4 is indedependent of p, q and u if $\rho<A^{-1}$. Combining these inequalities we obtain

$$N^{(p+1)m}(u)\leqq 9^mC\rho N^{pm}(Pu)+\frac{1}{2}N^{(p+1)m}(u)$$

$$+(9^mC+C_3+mC_4)\frac{((p+1)m)!}{(pm)!}N^{pm}(u)$$

$$+2mC_4\sum_{q=0}^{p-1}\frac{((p+1)m)!}{(qm)!}N^{qm}(u).$$

This proves the inequality (4).

Next we shall prove by induction that

$$(5)\qquad N^{pm}(u)\leqq C_1^p\sum_{q=0}^{p}\binom{p}{q}\frac{(pm)!}{(qm)!}N^0(P^qu)$$

with $C_1=C_0+1$.

When $p=0$, this is trivial. We assume that (5) is already proved when p is replaced by a smaller number. Write the inequality (4) and apply (4) to $N^{qm}(u)$ for $q=p-1, p-2, \cdots, 1$ successively. Then we easily obtain the estimate

$$N^{pm}(u) \leqq \sum_{q=0}^{p-1} \frac{(pm)!}{((q+1)m)!} C_1^{p-q} N^{qm}(Pu) + C_1^p (pm)! N^0(u).$$

Hence by induction hypotheses we have

$$\begin{aligned} N^{pm}(u) &\leqq C_1^p \sum_{q=0}^{p-1} \frac{(pm)!}{((q+1)m)!} \sum_{r=0}^{q} \binom{q}{r} \frac{(qm)!}{(rm)!} N^0(P^{r+1}u) + C_1^p (pm)! N^0(u) \\ &\leqq C_1^p \sum_{r=0}^{p-1} \sum_{q=r}^{p-1} \binom{q}{r} \frac{(pm)!}{((r+1)m)!} N^0(P^{r+1}u) + C_1^p (pm)! N^0(u) \\ &= C_1^p \sum_{r=0}^{p-1} \binom{p}{r+1} \frac{(pm)!}{((r+1)m)!} N^0(P^{r+1}u) + C_1^p (pm)! N^0(u). \end{aligned}$$

Now that (5) is established, it is easy to prove the analyticity of u. From (1) and (5) it follows that

$$\| \nabla^{pm} u \|_\delta \leqq \delta^{pm} C_1^p (B+1)^{p+1} (pm)! \qquad (p=0, 1, 2, \cdots).$$

By (3) we have

$$\| \nabla^{pm+t} u \|_{2\delta} \leqq C(\| \nabla^{(p+1)m} u \|_\delta + (1+\delta^{-m}) \| \nabla^{pm} \|_\delta)$$

for $t=1, \cdots, m-1$. Thus if B_1 is sufficiently large,

$$\| \nabla^q u \|_{2\delta} \leqq B_1^{q+1} (q+m)!$$

holds for all $q=0, 1, 2, \cdots$. Therefore u is analytic in $G_{2\delta}$.

Proof of Necessity. We shall prove by induction on p that the inequality

$$(6) \qquad \| \nabla^q P^p u \|_{L^2(G)} \leqq B^{q+pm+p+1} (q+pm)! \qquad (p, q=0, 1, 2, \cdots)$$

holds for B sufficiently large.

In case $p=0$, the analyticity of u implies the validity of (6) for all q with a constant B. Assume that (6) is true for a p and all q. Similarly to the proof of (4), applying Leibniz' formula to $\nabla^q P^{p+1} u = (\nabla^q P) P^p u$, we obtain

$$\begin{aligned} \| \nabla^q P^{p+1} u \| &\leqq \sum_{r=0}^{q} \frac{q!}{(q-r)!} A^{r+1} \sum_{s=0}^{m} \| \nabla^{q+s-r} P^p u \| \\ &\leqq (m+1) \sum_{r=0}^{q} \frac{q!}{(q-r)!} A^{r+1} \| \nabla^{q+m-r} P^p u \| + mq! A^{q+1} \sum_{s=0}^{m-1} \| \nabla^s P^p u \| \\ &\leqq \frac{(m+1)A}{B} \Bigg[\sum_{r=0}^{q} \frac{q!}{(q-r)!} \frac{(q+(p+1)m-r)!}{(q+(p+1)m)!} A^r B^{-r} \\ &\qquad + \sum_{s=0}^{m-1} \frac{q!(s+pm)!}{(q+(p+1)m)!} A^q B^{s-m-q} \Bigg] \times B^{q+(p+1)m+p+2} (q+(p+1)m)!. \end{aligned}$$

If B is so large that $AB^{-1} < 1/2$, the factor in the bracket does not exceed $m+2$ for any q. Therefore if we choose for B a number larger than $(m+2)^2 A$, then we have (6) with p replaced by $p+1$.

Remarks. This kind of theorem was first obtained by Nelson [4] in the form that the right hand side of (1) is replaced by $B^{p+1} p!$. The case of constant coefficients was treated by Komatsu [2, 3].

Some interesting applications to the theory of partial differential equations are given in [2] and Kotaké's forth-coming paper.

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References

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