16. *Time Change and Killing for Multi-Dimensional Reflecting Diffusion*

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1. Introduction. H. Tanaka and the author have defined in [8] *the local time on the boundary* for multi-dimensional reflecting diffusion. We show that this local time serves as a time change function in reducing the diffusion to *the Markov process on the boundary* introduced by T. Ueno [9]. This fact has been conjectured by him in [9]. In order to treat more general cases with killing (mass defect), we prove some general results on Markov processes. We also construct the diffusion with killing and sojourn on the boundary, which is an extension of the results obtained by K. Ito and H. P. McKean, Jr. [5] in one dimension and by N. Ikeda [3] in two dimensions.

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2. Definitions and notations. We use the definitions and notations of E. B. Dynkin [1] concerning Markov processes, unless specifically mentioned. Suppose a (temporally homogeneous) Markov process $X=(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ with state space $(E, \mathcal{B})$. We consider the following conditions:

M 1. E is a locally compact Hausdorff space with a countable base and $\mathcal{B}$ is the σ-algebra generated by the open sets.

M 2. $\mathbf{P}_x(\zeta>0)=1$ for all $x\in E$.

M 3. X is right-continuous.

M 4. X has the strict Markov property.

M 5. If $\tau_n(\omega)\uparrow\tau(\omega)<\zeta(\omega)$ for all $\omega\in B$, where τ_n are random variables independent of the future, then, for all x, $x_{\tau_n}\to x_\tau$ (a.e. $B, \mathbf{P}_x$).

M 6. $\mathcal{M}_{t+0}=\mathcal{M}_t$.

M 7. $\overline{\mathcal{M}}_t=\mathcal{M}_t$.[1]

If X satisfies the above conditions, X is a standard process in the sense of Dynkin. We call $\varphi_t(\omega)$ $(\omega\in\Omega_t)$ a *continuous [right-continuous] non-negative additive functional of* X, if it satisfies the following five conditions:

A 1. $\varphi_s(\omega)+\theta_s\varphi_t(\omega)=\varphi_{s+t}(\omega)$ for all $\omega\in\Omega_{s+t}$;

A 2. φ_t is $\mathcal{R}_t$-measurable;[2]

1) $\overline{\mathcal{M}}_t$ is the family of B such that, for every finite measure μ, there exist B_1 and $B_2\in\mathcal{M}_t$ satisfying $B_1\subseteq B\subseteq B_2$ and $\mathbf{P}_\mu(B_1)=\mathbf{P}_\mu(B_2)$.

2) We put $\mathcal{R}_t=\overline{\mathcal{N}_{t+0}\cap\mathcal{N}^*}$.

A 3. $0 \leqq \varphi_t(\omega) \leqq +\infty$.

A 4. $\mathbf{P}_x(\varphi_0=0)=1$ for all x.

A 5. $\varphi_t(\omega)$ is continuous [right-continuous] in t.

Often we impose the condition

A 6. $\varphi_t(\omega) < +\infty$.

We say that $\varphi_t(\omega)$ $(\omega \in \Omega_t)$ is a *continuous [right-continuous] non-negative additive functional of X in the wide sense,* if it satisfies A 1, 3, 4, 5, and

A 2′. φ_t is $\mathcal{M}_t \cap \mathcal{N}^*$ measurable.

Replace, in the definition of subprocess $\widetilde{X}$ in [1], $\mathcal{N}_t$-measurability of $\alpha_t(\omega)$ 3.21 Д by $\mathcal{M}_t$-measurability. Then we call $\widetilde{X}$ *subprocess of X in the wide sense.* The meaning of the *multiplicative functional in the wide sense* is obvious.

3. Time change and killing. Let $X=(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ be a Markov process with state space $(E, \mathcal{B})$ satisfying M 1—7, $\varphi_t(\omega)$ be a continuous non-negative additive functional of X, and $\psi_t(\omega)$ be a right-continuous non-negative additive functional of X satisfying A 6. Put $\tau_t(\omega)=\sup\{s: \varphi_s(\omega) \leqq t \text{ and } s<\zeta(\omega)\}$, $\zeta'(\omega)=\varphi_{\zeta-0}(\omega)$, $x'_t(\omega)=x_{\tau_t}(\omega)$ $(0 \leqq t < \zeta'(\omega))$, $\mathcal{M}'_t=\mathcal{M}_{\tau_t}$, $\theta'_t=\theta_{\tau_t}$, and $\mathbf{P}'_x=\mathbf{P}_x$, noting that τ_t is a random variable independent of the future for X. Put $E'=\{x: \mathbf{P}_x(\tau_0>0)=0\}$ and $\mathcal{B}'=\mathcal{B}[E']$. An almost similar argument to Volkonski [10] shows the following

THEOREM 3.1. *$X'=(x'_t, \zeta', \mathcal{M}'_t, \mathbf{P}'_x, \theta'_t)$ is a Markov process with state space $(E', \mathcal{B}')$ and satisfies* M 2, 3, 4, 6, and 7.[3] We say that X is transformed to X' through time change by φ_t.

Put $\psi'_t(\omega)=\psi_{\tau_t}(\omega)$. Then the following theorems hold.

THEOREM 3.2. *ψ'_t is a right-continuous non-negative additive functional of X' in the wide sense with the property* A 6.

THEOREM 3.3. *There exists the canonical subprocess of X' in the wide sense $Y^{(1)}=(y^{(1)}_t, \zeta^{(1)}, \mathcal{M}^{(1)}_t, \mathbf{P}^{(1)}_x, \theta^{(1)}_t)$ corresponding to the multiplicative functional in the wide sense* $\exp(-\psi'_t)$. *$Y^{(1)}$ satisfies* M 2, 3, 4, *and* 6.

In the construction of $Y^{(1)}$ from X, we have carried out killing after time change. Let us change the order of their operations. Let $X^C=(x^C_t, \zeta^C, \mathcal{M}^C_t, \mathbf{P}^C_x, \theta^C_t)$ be the canonical subprocess of X corresponding to the multiplicative functional $\exp(-\psi_t)$. θ^C_t is made to satisfy

(1) $\qquad \theta^C_t(\gamma^{-1}B, \zeta^C>0)=(\gamma^{-1}\theta_t B, \zeta^C>t)$ for all $B \in \mathcal{N}^*$.

Define $\widetilde{X}=(\tilde{x}_t, \tilde{\zeta}_t, \widetilde{\mathcal{M}}_t, \widetilde{\mathbf{P}}_x, \tilde{\theta}_t)$ where $\widetilde{\mathcal{M}}_t=\overline{\mathcal{M}^C_t}$ and other elements are the same as in X^C. Then we can prove

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3) In the case of standard process, we need not take a version in Theorem 1.4 of Volkonski [10] (communicated from M. Nagasawa).

THEOREM 3.4. *$\tilde{X}$ is a subprocess of X and satisfies* M 1—7.

The techniques essential to the proof are found in [1] and in the proof of Theorem 4.1 of Meyer [7].

Put $\tilde{\varphi}_t(\tilde{\omega})=\varphi_t(\gamma\tilde{\omega})$ for $\tilde{\omega}\in\tilde{\Omega}_t=\{\tilde{\zeta}>t\}$. Then, using (1), we have

THEOREM 3.5. *$\tilde{\varphi}_t$ is a continuous non-negative additive functional of $\tilde{X}$.*

Hence, by Theorems 3.1 and 3.4, we can transform $\tilde{X}$ to a new Markov process $Y^{(2)}=(y_t^{(2)},\zeta^{(2)},\mathcal{M}_t^{(2)},\mathbf{P}_x^{(2)},\theta_t^{(2)})$ through time change by $\tilde{\varphi}_t$. Of course $Y^{(2)}$ has a restricted state space and satisfies M 2, 3, 4, 6, and 7.

We are now in position to state the following

THEOREM 3.6.[4] *$Y^{(1)}$ and $Y^{(2)}$ have the same state space (E',B'). They are mutually equivalent and, in fact,*

$$\mathbf{M}_x^{(i)}\left[\int_0^{\zeta^{(i)}} e^{-\lambda t}f(y_t^{(i)})dt\right]=\mathbf{M}_x\left[\int_0^{\zeta} e^{-\lambda\varphi_t-\psi_t}f(x_t)d\varphi_t\right],\quad i=1,2,$$

for every bounded measurable function f.

If a Markov process Y is equivalent to $Y^{(1)}$ and $Y^{(2)}$, we call Y a process obtained from X through time change by φ_t and killing by ψ_t.

4. Reflecting diffusion and fundamental lemmas. Apart from general theory we turn to a special object, the reflecting diffusion. From now on we make all the assumptions in [8]. Thus D is a domain with compact closure $\bar{D}$ and sufficiently smooth boundary ∂D, and A is a second-order elliptic differential operator without non-differentiation term. From the reflecting A-diffusion $[x_t, W, \boldsymbol{B}, P_x]$ defined in [8], we get a Markov process $X=(x_t,+\infty,\mathcal{M}_t,\mathbf{P}_x,\theta_t)$ satisfying M 1—7, by putting $\mathcal{M}_t=\bar{\mathcal{N}}_{t+0}$ and $\theta_t B=\{w\colon w_t^+\in B\}$ (cf. [11]). The local time on the boundary $\mathfrak{t}_t(w)=\mathfrak{t}(t,w)$ defined in [8] is a continuous non-negative additive functional of X with the property A 6.

Suppose that $b(x)$ and $\beta(x)$ is non-negative functions in $C^{0,\mathrm{H}}(\bar{D})$[5] and $C^{2,\mathrm{H}}(\partial D)$, respectively. Then, we can prove the following two lemmas by the use of a lemma in [8] and the results of S. Ito [6].

LEMMA 4.1. *Given a function $f(x)$ in $C^{0,\mathrm{H}}(\bar{D})$, put*

$$u(t,x)=\mathbf{M}_x\left[\int_0^t e^{-\int_0^s b(x_r)dr-\int_0^s \beta(x_r)d\mathfrak{t}_r}f(x_s)ds\right].^{6)}$$

4) Special cases of this theorem are, though less explicite, found in [5], [3] and [2].

5) $C^{k,\mathrm{H}}(\bar{D})$ is the set of functions of $\bar{D}$ whose k-th order derivatives are uniformly Hölder continuous. $C^{0,\mathrm{H}}(\bar{D})$ is, especially, the set of uniformly Hölder continuous functions on $\bar{D}$.

6) $\mathfrak{t}_r(w)$ does not increase at r unless $x_r(w)\in\partial D$, so that $\int_0^s \beta(x_r)d\mathfrak{t}_r$ is determined by giving the function β on the boundary.

Then $u(t,x)$ *is the unique function satisfying* $\left(\frac{\partial}{\partial t}+b-A\right)u(t,x)=f(x)$ *on* D, $\left(\beta-\frac{\partial}{\partial n}\right)u(t,x)=0$ *on* ∂D, *and* $u(0,x)=0$.

LEMMA 4.2. *Given* $f(x)$ *in* $C^{0,\mathrm{H}}(\partial D)$, *put*

$$u(t,x)=\mathbf{M}_x\left[\int_0^t e^{-\int_0^s b(x_r)dr-\int_0^s \beta(x_r)dt_r} f(x_s)dt_s\right].$$

Then $u(t,x)$ *is the unique function satisfying* $\left(\frac{\partial}{\partial t}+b-A\right)u(t,x)=0$ *on* D, $\left(\beta-\frac{\partial}{\partial n}\right)u(t,x)=f(x)$ *on* ∂D, *and* $u(0,x)=0$.

5. Markov process on the boundary. Let $k\leqq 0$ be in $C^{0,\mathrm{H}}(\overline{D})$, $\alpha>0$ be in $C^{2,\mathrm{H}}(\partial D)$, and $\gamma\leqq 0$ be in $C^{2,\mathrm{H}}(\partial D)$. Let H_k denote the operator which transforms any continuous function f on the boundary to the solution u of $(A+k)u=0$ with boundary value f.[7] Put $\varphi_t=\int_0^t \alpha(x_s)dt_s$ and $\psi_t=\int_0^t |k(x_s)|ds+\int_0^t |\gamma(x_s)|\alpha(x_s)dt_s$. Obviously, φ_t and ψ_t are continuous additive functionals of X with the property A 6.

THEOREM 5.1.[8] *Let* Y *be a Markov process obtained from* X *through time change by* φ_t *and killing by* ψ_t. *Then the state space of* Y *is* ∂D, *and the transition operators of* Y *form a strongly continuous semigroup on* $C(\partial D)$ *with generator*[9] $\frac{1}{\alpha}\overline{\frac{\partial}{\partial n}H_k+\gamma}$.[10]

PROOF. Clearly the state space of Y is contained in ∂D. Let K_λ be the resolvent operator of Y. Then Theorem 3.6 implies that $K_\lambda f(x)=\mathbf{M}_x\left[\int_0^\infty e^{-\lambda\varphi_t-\psi_t}f(x_t)d\varphi_t\right]$. Applying Lemma 4.2 to the function $u(t,x)=\mathbf{M}_x\left[\int_0^t e^{-\lambda\varphi_s-\psi_s}f(x_s)d\varphi_s\right]$ and using the result of [6], we can show that $\left(\lambda-\gamma-\frac{1}{\alpha}\frac{\partial}{\partial n}H_k\right)K_\lambda f=f$ for $f\in C^{0,\mathrm{H}}(\partial D)$. Accordingly, by T. Ueno [9], K_λ is the resolvent operator of the strongly continuous semigroup on $C(\partial D)$ with generator $\frac{1}{\alpha}\overline{\frac{\partial}{\partial n}H_k+\gamma}$. This semigroup coincides with the transition semigroup of Y and the proof is complete.

In case both k and γ vanish, Y is non-cut-off. Hence the following

7) The existence and the uniqueness of such u are known (cf. [6]).

8) In the case of $k=0$, $\alpha=1$, and $\gamma=0$, the results of Theorems 5.1 and 5.2 were published in [4] in mimeographed form.

9) By the term of *generator*, we mean the infinitesimal generator in the sense of Hille-Yosida.

10) $\overline{\frac{\partial}{\partial n}H_k}$ is the smallest closed extention (in $C(\partial D)$) of $\frac{\partial}{\partial n}H_k$ restricted to $\{f: H_kf\in C^1(\overline{D})\}$.

THEOREM 5.2. $\mathbf{P}_x(\lim_{t\to+\infty} \mathfrak{t}_t=+\infty)=1$ *for all* $x\in\overline{D}$.

6. Diffusion with a certain boundary condition. Let k and γ be as in the previous section, $a>0$ be in $C^{0,\mathrm{H}}(\overline{D})$, and $\delta\leqq 0$ be in $C^{2,\mathrm{H}}(\partial D)$. Put $\hat{\varphi}_t=\int_0^t a(x_s)ds+\int_0^t|\delta(x_s)|\,dt_s$, $\hat{\psi}_t=\int_0^t|k(x_s)|\,a(x_s)ds+\int_0^t|\gamma(x_s)|\,dt_s$, and $\hat{A}=\frac{1}{a}A+k$. Then the following theorem can be proved.

THEOREM 6.1. *Let* Z *be a Markov process obtained from* X *through time change by* $\hat{\varphi}_t$ *and killing by* $\hat{\psi}_t$. *Then the transition operators of* Z *form a strongly continuous semigroup on* $C(\overline{D})$ *with generator* $\overline{\hat{A}}$ *restricted to* $\overline{\left\{u:\left(\frac{\partial}{\partial n}+\gamma+\delta\hat{A}\right)u=0\right\}}$.[11] *Moreover* $Z^{(2)}$ (cf. Section 3) *is a continuous Markov process with state space* $\overline{D}$ *and the properties* M 1—7.

The method of proof is similar to that of Theorem 5.1, but in this case we use both Lemmas 4.1 and 4.2. If G_λ is the resolvent operator of Z, $(\lambda-\hat{A})G_\lambda f=f$ and $\left(\frac{\partial}{\partial n}+\gamma+\delta\hat{A}\right)G_\lambda f=0$ for all $f\in C^{0,\mathrm{H}}(\overline{D})$. Strict increase of $\hat{\varphi}_t$ implies M 5 for $Z^{(2)}$.

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11) $\overline{\hat{A}}$ is the smallest closed extention of $\hat{A}$. $\hat{A}u(x)$ on ∂D is understood as $\lim_{y\in D,\,y\to x} Au(x)$. $\overline{\frac{\partial}{\partial n}+\gamma+\delta\hat{A}}$ is the extension of $\frac{\partial}{\partial n}+\gamma+\delta\hat{A}$ defined in [9].