15. A Note on Absolute Convergence of Fourier Series

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1. Theorems. Let $f(x)$ be integrable in Lebesgue sense in $(0, 2\pi)$, periodic with period 2π, and let

$$f(x)\sim\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx),$$

a_0 being, as we may, supposed to be zero. Then, its allied series is

$$\sum_{n=1}^{\infty}(a_n \sin nx-b_n \cos nx).$$

At $x=0$, these series become

$$\sum_{n=1}^{\infty}a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-b_n)$$

respectively.

In what follows, for the sake of convenience, $\sum_{n=1}^{\infty}$ will be sometimes denoted by $\sum$.

It is well known that f is bounded in $(0, 2\pi)$, and $a_n\geqq 0$ for all n, then $\sum a_n<\infty$. Cf. Paley [1]. But, the proposition $b_n\geqq 0$ for all n does not necessarily imply $\sum b_n<\infty$, unless some additional condition will be assumed concerning the function conjugate to f.

In this paper, we shall prove the following theorems.

THEOREM 1. If $f\in L$, and

$$f_h(0)=\frac{1}{2h}\int_0^h[f(t)+f(-t)]dt \tag{1.1}$$

is bounded for $0<h<\pi$, then the proposition $a_n\geqq 0$ for all n, or more generally

$$\sum_{n=1}^{\infty}(|a_n|-a_n)<\infty$$

implies

$$\sum_{n=1}^{\infty}|a_n|<\infty.$$

This theorem is clearly trivial when f is odd.

In the case $a_n\geqq 0$, this theorem is due to Szász [2, p. 697].

THEOREM 2. If $f\in L$, and

$$\bar{f}_h(0)=-\frac{1}{\pi}\int_h^\pi\frac{f(t)-f(-t)}{2\tan(t/2)}dt \tag{1.2}$$

is bounded for $0<h<\pi$, then

$$\sum_{n=1}^{\infty}(|b_n|-b_n)<\infty$$

implies

$$\sum_{n=1}^{\infty}|b_n|<\infty.$$

This theorem is trivial when f is even.

COROLLARY 1. Let by $\omega(\delta)$ denote the modulus of continuity of f in $(0, 2\pi)$, i.e.

$$\omega(\delta)=\omega(\delta; f; 0, 2\pi)=\sup|f(x+h)-f(x)|,$$

where sup is taken for all h, $|h|\leqq\delta$, and for all x and $x+h$ belonging to $(0, 2\pi)$, and let $\lambda(x)$ be any function such that $\lambda(n)>0$ for $n\geqq n_0$, and

$$\sum_{n=n_0}^{\infty}\frac{1}{n\lambda(n)}=\infty. \tag{1.3}$$

Under these circumstances, if

$$\sum_{n=n_0}^{\infty}\frac{\omega(1/n)}{n\lambda(n)}<\infty, \tag{1.4}$$

and

$$a_n>-\frac{\omega(1/n)}{n\lambda(n)} \quad \text{for } n\geqq n_0, \tag{1.5}$$

then we have

$$\sum_{n=1}^{\infty}|a_n|<\infty.$$

In this corollary, we may particularly take as $\lambda(x)$, e.g. $1/\log x$, 1, $\log x$, $\log x \log\log x$, etc. In the case $\lambda(x)=1/\log x$, this corollary is due to Tomić [3].

COROLLARY 2. If $\omega(\delta)=\omega(\delta; f; 0, 2\pi)$,

$$\sum_{n=1}^{\infty}\frac{\omega(1/n)}{n}<\infty, \tag{1.6}$$

and

$$a_n>-\frac{\omega(1/n)}{n},\quad b_n>-\frac{\omega(1/n)}{n},\quad \text{for } n\geqq 1,$$

then the Fourier series of f converges absolutely everywhere.

(N. B. 1) We notice that the single condition (1.6) implies the uniform convergence in $(0, 2\pi)$ of the Fourier series of f and its allied series, since (1.6) implies $\omega(1/n)=o(1/\log n)$ by Lemma 4 below, and the continuity in $(0, 2\pi)$ of the function conjugate to f.

2. Proofs of Theorems. We need some lemmas. It is known that if there exists the limit of $f_h(0)$ for $h\to+0$, then $\sum a_n$ is summable A to this limit, see Zygmund [4, p. 101 (7.9)], and that if there exists the limit of $\bar{f}_h(0)$ for $h\to+0$, then $\sum(-b_n)$ is summable A to this limit, see also loc. cit. [4, p. 104]. And indeed in both cases the summability A may be, as it is easily shown, replaced by the summability $(C, 2)$. Quite analogously, one obtains the following two lemmas.

LEMMA 1. If $f\in L$, and $f_h(0)$ defined in (1.1) is bounded, for $0<h$

$<\pi$, then $\sum a_n$ is bounded in Abel sense.

LEMMA 2. If $f\in L$, and $\bar{f}_h(0)$ defined in (1.2) is bounded for $0<h<\pi$, then $\sum b_n$ is bounded in Abel sense.

Let us denote by K, in the sequel, an absolute positive constant which may not be the same in different occurrences.

LEMMA 3. If the real series $\sum u_n$ is bounded in Abel sense, i.e.

$$|\sum_{n=1}^{\infty} u_n r^n| < K \tag{2.1}$$

holds for every value of r such that $0<r<1$, and if

$$\sum_{n=1}^{\infty}(|u_n|-u_n)<\infty, \tag{2.2}$$

then we have

$$\sum_{n=1}^{\infty}|u_n|<\infty.$$

PROOF. From (2.2) one obtains, for $0<r<1$,

$$\sum_{n=1}^{\infty}(|u_n|-u_n)r^n<K$$

which together with (2.1) yields

$$\sum_{n=1}^{\infty}|u_n|r^n<2K,$$

and then

$$\sum_{n=1}^{N}|u_n|r^n<2K$$

for every positive integer N. Making $r\to 1-0$ and then $N\to\infty$, we have successively

$$\sum_{n=1}^{N}|u_n|\leqq 2K,$$

$$\sum_{n=1}^{\infty}|u_n|\leqq 2K,$$

which is the required.

LEMMA 4. Let v_n decrease to zero with $1/n$, and $\lambda(x)$ be any function such that $\lambda(n)>0$ for $n\geqq n_0$, and

$$\sum_{n=n_0}^{N}\frac{1}{n\lambda(n)}=\mu(N)\to\infty \tag{2.3}$$

as $N\to\infty$. Then

$$\sum_{n=n_0}^{\infty}\frac{v_n}{n\lambda(n)}<\infty \tag{2.4}$$

implies

$$v_n=o(1/\mu(n)) \quad \text{as } n\to\infty.$$

In particular, letting $\lambda(x)=1$,

$$\sum_{n=1}^{\infty}\frac{v_n}{n}<\infty$$

implies

$$v_n=o(1/\log n) \quad \text{as } n\to\infty,$$

and letting $\lambda(x)=\log x$,

$$\sum_{n=2}^{\infty}\frac{v_n}{n\log n}<\infty$$

implies

$$v_n=o(1/\log\log n) \quad \text{as } n\to\infty.$$

(N. B. 2) If we put $x\lambda(x)=1$ in the lemma, then we have the classical result that if $v_n\downarrow 0$ and $\sum v_n<\infty$ then $v_n=o(1/n)$ as $n\to\infty$.

PROOF. From (2.4), we see that for any positive ε there exists an integer m such that

$$\sum_{n=m}^{N}\frac{v_n}{n\lambda(n)}<\frac{\varepsilon}{2} \tag{2.5}$$

holds for every $N\geqq m$. And, by (2.3) we can choose N_1 so large that for the above fixed m

$$\sum_{n=m}^{N}\frac{1}{n\lambda(n)}>\frac{1}{2}\mu(N) \tag{2.6}$$

holds for all $N\geqq N_1$. On the other hand, since $\omega(1/n)$ decreases with $1/n$, it holds

$$\sum_{n=m}^{N}\frac{v_n}{n\lambda(n)}>v_N\sum_{n=m}^{N}\frac{1}{n\lambda(n)}.$$

Hence, for every number N satisfying (2.6) one obtains

$$\frac{\varepsilon}{2}>v_N\frac{1}{2}\mu(N), \text{ i.e. } v_N<\frac{\varepsilon}{\mu(N)},$$

which proves the lemma.

We now prove the theorems.

Theorem 1 follows immediately from Lemmas 1 and 3, and Theorem 2 does from Lemmas 2 and 3.

PROOF OF COROLLARY 1. By Lemma 4, (1.4) together with (1.3) yields $\omega(1/n)\to 0$ for $n\to\infty$, which a fortiori implies the existence of the finite limit of $f_h(0)$ for $h\to+0$, and clearly (1.5) together with (1.4) yields $\sum(|a_n|-a_n)<\infty$. So, the corollary follows from Theorem 1.

PROOF OF COROLLARY 2. $\sum|a_n|<\infty$ is a result from Corollary 1 with $\lambda(x)=1$. Next, observing that $\sum n^{-1}\omega(1/n)<\infty$ is equivalent to $\int_0^{\pi}t^{-1}\omega(t)dt<\infty$ which implies the existence of the finite limit of $\bar{f}_h(0)$ for $h\to+0$, $\sum|b_n|<\infty$ is a result from Theorem 2. Thus, we get the corollary.

References

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