36. On the Absolute Nörlund Summability Factors of a Fourier Series

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1.1. *Definitions.* Let $\sum u_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \cdots + p_n; \quad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation:

$$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu = \frac{1}{P_n}\sum_{\nu=0}^{n} P_{n-\nu}u_\nu, \quad (P_n \neq 0), \tag{1.1.1.}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum u_n$ is said to be summable (N, p_n) to the sum s if $\lim_{n\to\infty} t_n$ exists and is equal to s, and is said to be absolutely summable (N, p_n), or $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation,[1] that is, the series $\sum |t_n - t_{n-1}|$ is convergent. In the special case in which

$$p_n = 1/(n+1) \tag{1.1.2}$$

the Nörlund mean reduces to the Harmonic mean.

Thus summability $|N, p_n|$, where p_n is defined by (1.1.2) is the same as the absolute Harmonic summability.

1.2. Let $f(t)$ be a periodic function, with period 2π, and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Then the Fourier series of $f(t)$ is

$$\sum (a_n \cos nt + b_n \sin nt) = \sum A_n(t). \tag{1.2.1}$$

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

$\tau = [1/t]$, i.e., the greatest integer contained in $1/t$.

K = an absolute constant, not necessarily the same at each occurrence.

2.1. We establish the following theorem.

Theorem. *If $\phi(t) \in BV\,(0, \pi)$, and $\{\lambda'_n\}$, where $\lambda'_n = \frac{\lambda_n}{n}$, is monotonic increasing then $\sum_{n=1}^{\infty} nA_n(t)/\lambda_n$ is summable $|N, p_n|$, provided $\{p_n\}$ satisfies the following conditions:*

(i) *$\{p_n\}$ is monotomic diminishing, and P_n is monotonic in-*

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1) Symbolically, $\{t_n\} \in BV$; similarly by '$f(x) \in BV\,(h, k)$' we shall mean that $f(x)$ is a function of bounded variation over the interval (h, k).

creasing, tending to ∞ *with* n;

(ii) *there exists a monotonic increasing function of* n, μ_n *say*, $\mu_n=\mu_n^*+1$ ($<n-1$, *for sufficiently large* n), *such that*

(a) $P_n-P_k=O(1)$, for $k>[\mu_n^*]$, *as* $n\to\infty$;

(b) $\dfrac{P_n}{p_n}=O(\lambda_{n-[\mu_n^*]})$;

(c) $\sum_n^\infty \dfrac{1}{P_nP_{n-1}} \sum_{[\mu_n]\le k<n} p_k/\lambda_{n-k}$ *is convergent*;

(d) $\sum_{[\mu_n]\le k<n} |\Delta\{P_k(p_k-p_n\}/\lambda_{n-k}\}|=O(P_n^2/n^2)$,

(e) $\sum_{0\le k<\mu_n^*}\left|\Delta\left\{\left(\dfrac{P_n}{p_n}-\dfrac{P_k}{p_k}\right)\dfrac{1}{\lambda_{n-k}}\right\}\right|=O(1)$, as $n\to\infty$.

It may be remarked that the following theorem due to Varshney follows from our theorem in the case in which $\lambda_n=n\log(n+1)$ and $p_n=1/(n+1)$.

Theorem A.[2] If $\phi(t)\in BV\,(0,\pi)$ then the series $\sum A_n(t)/\log(n+1)$ is absolutely summable by Harmonic means.

2.2. We require the following lemmas for the proof of the theorem.

Lemma 1.[3] *If* p_n *is non-negative and non-increasing, then, for* $0\le a\le b\le\infty$, $0\le t\le\pi$, *and any* n, *we have*

$$\left|\sum_{k=a}^{b} p_k \sin(n-k)t\right|\le KP_\tau .$$

Lemma 2. *For any integers* a *and* b, *we have*

$$\sum_{n=a}^{b}\sin nt=O(1/t).$$

Lemma 3. *If* $P_n\to\infty$, *as* $n\to\infty$, *then*

$$\sum_{m+1}^{\infty}\frac{p_n}{P_nP_{n-1}}=O(1/P_m)$$

as $m\to\infty$.

The proofs of Lemmas 2 and 3 are easy.

2.3. *Proof of the theorem.* Writing $u_\nu=\nu A_\nu(t)/\lambda_\nu$, and

$$t_n=\frac{1}{P_n}\sum_{\nu=0}^{n}P_{n-\nu}u_\nu,$$

we have

$$\begin{aligned} t_n-t_{n-1}&=\sum_{\nu=0}^{n-1}\left(\frac{P_\nu}{P_n}-\frac{P_{\nu-1}}{P_{n-1}}\right)u_{n-\nu}\\ &=\frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}(P_np_\nu-P_\nu p_n)u_{n-\nu}. \end{aligned}$$

Now, since

2) Varshney [2].

3) McFadden [1].

$$A_n(t)=\frac{2}{\pi}\int_0^\pi \phi(t)\cos nt\,dt,$$

$$t_n-t_{n-1}=\frac{2}{\pi}\int_0^\pi \phi(t)\left(\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\frac{(n-k)\cos(n-k)t}{\lambda_{n-k}}\right)dt.$$

Thus, in order to prove the theorem, we have to show that

$$\sum_n\left|\int_0^\pi \phi(t)g(n,t)\,dt\right|<\infty,$$

where

$$g(n,t)=\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\frac{(n-k)\cos(n-k)t}{\lambda_{n-k}}.$$

We observe that

$$\int_0^\pi \phi(t)g(n,t)\,dt=-\int_0^\pi\left(\int_0^t g(n,u)\,du\right)d\phi(t)$$

and

$$\sum_n\left|\int_0^\pi\left(\int_0^t g(n,u)\,du\right)d\phi(t)\right|\le\int_0^\pi|d\phi(t)|\left\{\left|\sum_n\left(\int_0^t g(n,u)\,du\right)\right|\right\}.$$

But by hypothesis $\int_0^\pi|d\phi(t)|<\infty$. Thus it is enough to show that, uniformly in $0<t\le\pi$,

$$\sum=\sum_n\left|\int_0^t g(n,u)\,du\right|=O(1).$$

We have

$$\begin{aligned}\sum&=\sum_n\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\frac{\sin(n-k)t}{\lambda_{n-k}}\right|\\&\le\sum_1^\tau\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\frac{\sin(n-k)t}{\lambda_{n-k}}\right|\\&\quad+\sum_{\tau+1}^\infty\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{[\mu_n^*]}(P_np_k-p_nP_k)\frac{\sin(n-k)t}{\lambda_{n-k}}\right|\\&\quad+\sum_{\tau+1}^\infty\frac{1}{P_nP_{n-1}}\left|\sum_{k=[\mu n]}^{n-1}(P_np_k-p_nP_k)\frac{\sin(n-k)t}{\lambda_{n-k}}\right|\\&=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3,\text{ say.}\end{aligned}$$

Now since

$$|\sin(n-k)t|\le(n-k)t,$$

and on account of the hypothesis (i), $P_np_k\ge P_kp_n$ for $k\le n$, we have

$$\begin{aligned}\sum\nolimits_1&=\sum_1^\tau\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\frac{\sin(n-k)t}{(n-k)\lambda'_{n-k}}\right|\\&\le K\cdot\frac{1}{\lambda'_1}\cdot t\sum_1^\tau\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}p_kP_n\\&\le Kt\cdot\sum_1^\tau 1\\&\le K.\end{aligned}$$

Applying Abel's transformation, we get

$$\begin{aligned}\sum\nolimits_2 &= \sum_{\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\left|\sum_{k=0}^{[\mu_n^*]}\left(\frac{P_n}{p_n}-\frac{P_k}{p_k}\right)\frac{p_k}{\lambda_{n-k}}\sin(n-k)t\right| \\
&\le \sum_{\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\left|\sum_{k=0}^{[\mu_n^*]-1}\Delta\left\{\left(\frac{P_n}{p_n}-\frac{P_k}{p_k}\right)\frac{1}{\lambda_{n-k}}\right\}\sum_{\kappa=0}^{k}p_\kappa\sin(n-\kappa)t\right| \\
&\quad + \sum_{\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\left|\left(\frac{P_n}{p_n}-\frac{P_{[\mu_n^*]}}{p_{[\mu_n^*]}}\right)\frac{1}{\lambda_{n-[\mu_n^*]}}\sum_{k=0}^{[\mu_n^*]}p_k\sin(n-k)t\right| \\
&\le KP_\tau\sum_{\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{[\mu_n^*]-1}\left|\Delta\left\{\left(\frac{P_n}{p_n}-\frac{P_k}{p_k}\right)\frac{1}{\lambda_{n-k}}\right\}\right| \\
&\quad + KP_\tau\sum_{\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\left(\frac{P_n}{p_n}-\frac{P_{[\mu_n^*]}}{p_{[\mu_n^*]}}\right)\frac{1}{\lambda_{n-[\mu_n^*]}} \\
&\le KP_\tau\sum_{\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}+KP_\tau\sum_{\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\frac{P_n}{p_n}\frac{1}{\lambda_{n-[\mu_n^*]}} \\
&\le K,\end{aligned}$$

by virtue of hypotheses (e), (b), and lemma 3.

Now we proceed to show that $\sum_3=O(1)$.

We have

$$\begin{aligned}\sum\nolimits_3 &\le \sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{k=[\mu n]}^{n-1}(P_n-P_k)p_k\frac{\sin(n-k)t}{\lambda_{n-k}}\right| \\
&\quad + \sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{k=[\mu n]}^{n-1}P_k(p_k-p_n)\frac{\sin(n-k)t}{\lambda_{n-k}}\right| \\
&= \sum\nolimits_{31}+\sum\nolimits_{32},\ \text{say.}\end{aligned}$$

Now

$$\begin{aligned}\sum\nolimits_{31} &= \sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{k=[\mu n]}^{n-1}(P_n-P_k)\sin(n-k)t\frac{p_k}{\lambda_{n-k}}\right| \\
&\le K\sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left[\sum_{k=[\mu n]}^{n-1}\frac{p_k}{\lambda_{n-k}}\right] \\
&\le K,\end{aligned}$$

by hypotheses (a) and (c).

Applying Abel's transformation to the inner sum in we have, by Lemma 2,

$$\begin{aligned}\sum\nolimits_{32} &\le \sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{k=[\mu n]}^{n-2}\Delta\left\{\frac{P_k(p_k-p_n)}{\lambda_{n-k}}\right\}\sum_{\kappa=0}^{k}\sin(n-\kappa)t\right| \\
&\quad + \sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\frac{P_{n-1}(p_{n-1}-p_n)}{\lambda_1}\sum_{k=0}^{n-1}\sin(n-k)t\right| \\
&\leqq K\tau\sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left[\sum_{k=[\mu n]}^{n-2}\left|\Delta\left\{\frac{P_k(p_k-p_n)}{\lambda_{n-k}}\right\}\right|\right] \\
&\quad + K\tau\sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\cdot P_{n-1}(p_{n-1}-p_n) \\
&= \sum\nolimits_{321}+\sum\nolimits_{322},\ \text{say.}\end{aligned}$$

Also, by hypothesis (d),

$$\sum\nolimits_{321}\le K\tau\sum_{\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\frac{P_n^2}{n^2}$$

$$\leqq K\tau \sum_{\tau+1}^{\infty} \frac{1}{n^2}$$
$$\leq K.$$

Now since p_n is monotonic decreasing, while P_n is monotonic increasing, $p_n/P_n < p_{n-1}/P_{n-1}$, so that p_n/P_n is monotonic decreasing, and $np_n \leq P_n$.

Hence

$$\begin{aligned}\sum\nolimits_{322} &= K\tau \sum_{\tau+1}^{\infty} \left(\frac{p_{n-1}}{P_n} - \frac{p_n}{P_n}\right) \\ &< K\tau \sum_{\tau+1}^{\infty} \left(\frac{p_{n-1}}{P_{n-1}} - \frac{p_n}{P_n}\right) \\ &\leqq K\tau p_\tau / P_\tau \\ &\leqq K.\end{aligned}$$

This completes the proof of our theorem.

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References

[1] McFadden, Leonard: Absolute Nörlund summability, Duke Math. Jour., **9**, 543–568 (1942).

[2] Varshney, O. P.: On the absolute Harmonic summability of a series related to a Fourier series, Proc. Amer. Math. Soc., **10**, 784–789 (1959).