47. Some Properties of Completely Normal and Collectionwise Normal Spaces

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1. In our previous note [4] we have proved the following theorem.

Theorem 1. *If for any locally finite family* $\{X_\alpha \mid \alpha \in \Omega\}$ *of closed subsets of a topological space* R *there exists a locally finite covering* $\{H_\alpha \mid \alpha \in \Omega\}$ *of closed subsets of* R *satisfying one of the following equivalent conditions* (A), (B), *and* (C), *then* R *is completely normal and collectionwise normal:*

(A) $H_\alpha \cap H_\beta \cap (X_\alpha \cup X_\beta) = X_\alpha \cap X_\beta$, $(\alpha, \beta \in \Omega)$

(B) $H_\alpha \cap H_\beta = X_\alpha \cap X_\beta$, $(\alpha, \beta \in \Omega)$

(C) $H_\alpha \cap (\bigcup_{\gamma \in \Omega} X_\gamma) = X_\alpha$ $(\alpha \in \Omega)$.

In this paper, in connection with the above theorem we shall establish necessary and sufficient conditions for topological spaces to be completely normal and collectionwise normal (see Theorems 3 and 4).

2. We shall first prove the following theorem.

Theorem 2. *Let* R *be a completely normal space. If* $\{X_\alpha \mid \alpha \in \Omega\}$ *is a family of closed subsets of* R *and* $\{U_\alpha \mid \alpha \in \Omega\}$ *is a locally finite family of open subsets of* R *such that* $X_\alpha \subset U_\alpha$ *for each* $\alpha \in \Omega$, *then there exists a locally finite closed covering* $\{H_\alpha \mid \alpha \in \Omega\}$ *of* R *satisfying one of the equivalent conditions* (A), (B), *and* (C) *of Theorem 1.*

Proof. If Ω is a finite set, this theorem has already been established (see [4], Theorem 1), so we assume that Ω is infinite. Now, we shall construct a family $\{H_\alpha \mid \alpha \in \Omega\}$ of closed subsets of R satisfying the condition (C) of Theorem 1 by transfinite induction.

Let η be a limit ordinal number such that $\Omega = \{\alpha \mid \alpha < \eta\}$ and let us put $U = \bigcup_{\alpha \in \Omega} U_\alpha$, $U'_\mu = \bigcup_{\alpha > \mu} U_\alpha$, $X = \bigcup_{\alpha \in \Omega} X_\alpha$, $X'_\mu = \bigcup_{\alpha > \mu} X_\alpha$. Let ν be an ordinal number such that $\nu < \eta$. We assume that to every $\mu < \nu$ there exist two closed subsets F_μ, F'_μ of U satisfying the following conditions:

$$
(P_\mu) \quad \begin{cases} (1) & U_\gamma \supset F_\gamma, \quad (\gamma \leqq \mu); \quad U'_\mu \supset F'_\mu, \\ (2) & (\bigcup_{\gamma \leqq \mu} F_\gamma) \cup F'_\mu = U, \\ (3) & F_\gamma \cap X = X_\gamma \qquad (\gamma \leqq \mu), \\ (4) & F'_\mu \cap X = X'_\mu. \end{cases}
$$

Then we shall construct two closed subsets F_ν, F'_ν of U satisfying the condition (P_ν).

We shall first show that the relation $\bigcap_{\mu < \nu} \bigcup_{\alpha > \mu} U_\alpha = \bigcup_{\mu \geqq \nu} U_\mu$ holds. It is evident that $\bigcap_{\mu < \nu} \bigcup_{\alpha > \mu} U_\alpha \supset \bigcup_{\mu \geqq \nu} U_\mu$. So conversely, let $x (\in U)$ be any point not contained in $\bigcup_{\mu\geqq\nu} U_\mu$. Since $\{U_\alpha \mid \alpha\in\Omega\}$ is locally finite there exists the greatest index $\kappa(<\nu)$ such that U_κ contains x. Hence x is not contained in $\bigcup_{\alpha>\kappa} U_\alpha$. Consequently, x is not contained in $\bigcap_{\mu<\nu}\bigcup_{\alpha<\mu} U_\alpha$. Put $F''_\nu=\bigcap_{\mu<\nu} F'_\mu$. It is also shown similarly that $F''_\nu \cap X=\bigcup_{\mu\geqq\nu} X_\mu$.

Next, we shall construct $\{F_\mu \mid \mu\leqq\nu\}$ and F'_ν. Then by $\bigcap_{\mu<\nu} U'_\mu=\bigcup_{\mu\geqq\nu} U_\mu$ we have $F''_\nu\subset\bigcup_{\mu\geqq\nu} U_\mu$. By the assumption of (P_μ), (4) we can see that X_ν and X'_ν are contained in F''_ν. Since $\{(U_\nu-U'_\nu)\cup(X_\nu-X'_\nu)\}\cap F''_\nu$ and $\{(U'_\nu-U_\nu)\cup(X'_\nu-X_\nu)\}\cap F''_\nu$ are separated subsets of completely normal space F''_ν, there exist two disjoint open subsets W, W' of F''_ν such that $W\supset\{(U_\nu-U'_\nu)\cup(X_\nu-X'_\nu)\}\cap F''_\nu$ and $W'\supset\{(U'_\nu-U_\nu)\cup(X'_\nu-X_\nu)\}\cap F''_\nu$. If we put $F_\nu=(F''_\nu-W')\cup X_\nu$ and $F'_\nu=(F''_\nu-W)\cup X'_\nu$, then F_ν and F'_ν are closed subsets of U.

We shall prove that $\{F_\mu \mid \mu\leqq\nu\}$ and F'_ν satisfy the conditions of (P_ν). Since $F''_\nu\subset U_\nu\cup U'_\nu$, we have

$$\begin{aligned}F''_\nu-W'&\subset F''_\nu-\{(U'_\nu-U_\nu)\cup(X'_\nu-X_\nu)\}\cap F''_\nu\\&\subset F''_\nu-(U'_\nu-U_\nu)\\&\subset U_\nu.\end{aligned}$$

By the assumption of the theorem we have $X_\nu\subset U_\nu$. Hence, $U_\nu\supset(F''_\nu-W')\cup X_\nu=F_\nu$ is concluded. Similarly, $U'_\nu\supset F'_\nu$ is obtained. Thus, (P_ν), (1) is satisfied. We have (P_ν), (2) by relations $(\bigcup_{\mu<\nu} F_\mu)\cup F''_\nu=U$ and $F_\nu\cup F'_\nu=F''_\nu$. Since

$$\begin{aligned}F_\nu\cap X&=\{(F''_\nu-W')\cap X\}\cup(X_\nu\cap X)\\&\subset[\{F''_\nu-(X'_\nu-X_\nu)\}\cap X]\cup X_\nu\\&=\{F''_\nu\cap X-(X'_\nu-X_\nu)\cap X\}\cup X_\nu\\&=\{\textstyle\bigcup_{\mu\geqq\nu} X_\mu-(X'_\nu-X_\nu)\}\cup X_\nu\\&=X_\nu\end{aligned}$$

and $F_\nu\cap X\supset X_\nu$, we have $F_\nu\cap X=X_\nu$, that is, (P_ν), (3). Finally, we have (P_ν), (4) using the following relations

$$\begin{aligned}F'_\nu\cap X&\subset\{(F''_\nu-W)\cap X\}\cup(X'_\nu\cap X)\\&\subset\{F''_\nu\cap X-(X_\nu-X'_\nu)\cap X\}\cup X'_\nu\\&=\{\textstyle\bigcup_{\mu\geqq\nu} X_\mu-(X_\nu-X'_\nu)\}\cup X'_\nu\\&=X'_\nu\end{aligned}$$

and $F'_\nu\cap X\supset X'_\nu$.

Since $|\Omega|\geqq\aleph_0$ and $\{U_\alpha \mid \alpha\in\Omega\}$ is locally finite, $\{F_\alpha \mid \alpha\in\Omega\}$ is a covering of U. As X is a closed subset of the normal space R, there exists an open set G such that $X\subset G\subset\overline{G}\subset U$. If we put $H_1=(F_1\cap\overline{G})\cup(R-G)$ and $H_\alpha=F_\alpha\cap\overline{G}(1<\alpha<\eta)$, then $\{H_\alpha \mid \alpha\in\Omega\}$ is a locally finite closed covering of R satisfying the condition (C) of Theorem 1, q.e.d.

3. We shall now prove our main theorem.

Theorem 3. *In order that a topological space R be completely normal and collectionwise normal, it is necessary and sufficient that for any locally finite and order finite family $\{X_\alpha \mid \alpha\in\Omega\}$ of closed subsets of R there exists a locally finite closed covering $\{H_\alpha \mid \alpha\in\Omega\}$ of*

R satisfing one of the equivalent conditions (A), (B), *and* (C) *of Theorem 1.*

Theorem 4. *Let R be a topological space having the following property: For any locally finite collection $\{X_\alpha \mid \alpha \in \Omega\}$ of closed subsets of R there exists a locally finite open covering $\{W_\alpha \mid \alpha \in \Omega\}$ of $\bigcup_{\alpha \in \Omega} X_\alpha$ such that $X_\alpha \subset W_\alpha$ for each $\alpha \in \Omega$. In order that R be a completely normal and collectionwise normal space it is necessary and sufficient that there exists a locally finite closed covering $\{H_\alpha \mid \alpha \in \Omega\}$ of R satisfying one of the equivalent conditions* (A), (B), *and* (C) *of Theorem 1.*

These theroems are easily proved by the above Theorems 1, 2, and the following lemma:

Lemma (Katětov [5]). *For a normal space R each of the following properties is equivalent to collectionwise normality:*

(a) *If $\{X_\alpha\}$ is a closed locally finite family in R such that for some positive integer n the intersection of any $n+1$ members of $\{X_\alpha\}$ is empty, then there exists an open locally finite family $\{U_\alpha\}$ such that $X_\alpha \subset U_\alpha$.*

(b) *For any closed subset X of R, if $\{X_\alpha\}$ [respectively, $\{W_\alpha\}$] is a closed [respectively, open] locally finite family for the relative topology of X such that $X_\alpha \subset W_\alpha$, then there exists an open locally finite family $\{U_\alpha\}$ in R such that $X_\alpha \subset U_\alpha \frown X \subset W_\alpha$.*

By a result of Katětov [5] and Theorem 2 we have the following:

Theorem 5. *Let R be countably paracompact, completely normal and collectionwise normal. Then for any locally finite family $\{X_\alpha \mid \alpha \in \Omega\}$ of closed subsets of R there exists a locally finite closed covering $\{H_\alpha \mid \alpha \in \Omega\}$ of R satisfying one of the equivalent conditions* (A), (B), *and* (C) *of Theorem 1.*

Remark 1. If R is perfectly normal and collectionwise normal, then the above theorem is also true by a result of Dowker [2].

Remark 2. In case R is a fully normal and completely normal space, the above theorem is easily deduced from the construction in the proof of Theorem 3 [3].

References

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