46. A Remark on General Imbedding Theorems in Dimension Theory

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Once we have constructed [1] a universal n-dimensional set for *general* metric spaces which is a rather complicated subset of C. H. Dowker's generalized Hilbert space. In this brief note we shall show that we can find a simpler universal n-dimensional set in a countable product of H. J. Kowalsky's star-spaces.

On the other hand, we have found [2] in the product of a generalized Baire 0-dimensional space and the Hilbert-cube a universal countable-dimensional set for metric spaces with a σ-star-finite basis. A universal countable-dimensional set for *general* metric space will also be found in such a product of star spaces.

Let E_α, $\alpha\in A$ be a system of unit segments $[0, 1]$. By identifying all zeros in $\bigcup\{E_\alpha \mid \alpha\in A\}$ we get a set S. We introduce a metric in S as follows.

$$\begin{aligned}\rho(x, y) &= |x-y| \quad \text{if } x, y \text{ belong to the same segment,}\\ &= |x|+|y| \quad \text{if } x, y \text{ belong to the distinct segments.}\end{aligned}$$

Then we get a metric space S called the *star-space* with the index set A. H. J. Kowalsky [3] proved that

a topological space R is metrizable if and only if it can be imbedded in a countable product of star-spaces.

Now we can assert the following theorem in dimension theory.

Theorem 1. *A metric space R has* (*covering*) *dimension $\leqq n$ if and only if it can be imbedded in the subset K_n of a countable product P of star-spaces, where we denote by K_n the set of points in P at most n of whose non-vanishing coordinates are rational.*

Proof. To see $\dim K_n \leqq n$ we decompose K_n as $K_n=\bigcup_{i=0}^{n} K_i'$ for the sets K_i' of points in P just i of whose non-vanishing coordinates are rational. We consider a given class a_j, $j=1\cdots i$ of i rational numbers with $0<a_j\leqq 1$. Then the set of points in K_i' whose j-th coordinates are equal to a_j is a 0-dimensional closed subset of K_i'. This assertion is proved by the product theorem in dimension theory since it is easily seen that the set of irrational points and zero in a star-space has dimension 0. Hence it follows from the sum-theorem that K_i' as a countable sum of such closed sets is 0-dimensional. This implies by the decomposition theorem that $\dim K_n \leqq n$.

Conversely, we suppose R is a general metric space with dim $R \leqq n$. By Bing's metrization theorem [4], there exists a σ-discrete open basis

$$\mathfrak{W}_m = \{W_{m\alpha} \mid \alpha \in A_m\},\ m=1, 2, \cdots.$$

We can assume without loss of generality that there exist open sets $V_{m\alpha}, \alpha \in A_m,\ m=1, 2, \cdots$ such that

$$\bar{V}_{m\alpha} = F_{m\alpha} \subset W_{m\alpha},$$

for every neighborhood $U(p)$ of every point p of R there exists m and $\alpha \in A_m$ for which

$$p \in F_{m\alpha} \subset W_{m\alpha} \subset U(p).$$

Putting

$$W_m = \bigcup \{W_{m\alpha} \mid \alpha \in A_m\},$$
$$F_m = \bigcup \{F_{m\alpha} \mid \alpha \in A_m\},$$

we get open sets W_m and closed sets F_m satisfying $F_m \subset W_m$.

Now we decompose R by the decomposition theorem as $R = \bigcup_{k=1}^{n+1} R_k$ for 0-dimensional sets R_k. Then we define open sets $U_{mr},\ m=1, 2, \cdots$; r: rational numbers with $0 < r < \sqrt{2}/2m$ such that

(1) $F_m \subset U_{mr} \subset \bar{U}_{mr} \subset U_{mr'} \subset \bar{U}_{mr'} \subset W_m$ if $r > r'$,

(2) $\bar{U}_{mr} = \bigcap \{U_{mr'} \mid r' < r\},\ U_{mr} = \bigcup \{\bar{U}_{mr'} \mid r' > r\}$,

(3) order $_p\{B(U_{mr}) \mid m=1, 2, \cdots,\ 0 < r < \sqrt{2}/2m,\ r\text{: rational}\} \leqq k-1$ for each point $p \in R_k$,

where $B(U)$ denotes the boundary of a set U, and order $_p\mathfrak{U}$ denotes the number of elements of a collection $\mathfrak{U}$ which contain p. The process to construct U_{mr} is quite parallel to that in the proof of Lemma 4.1 [2], so it will not be mentioned here.

Let $U^m_{r\alpha} = U_{mr} \cap W_{m\alpha}$; then, since each $\{W_{m\alpha} \mid \alpha \in A_m\}$ is discrete, $U^m_{r\alpha},\ m=1, 2, \cdots, \alpha \in A_m, 0 < r < \sqrt{2}/2m$, r: rational are open sets such that

$$F_{m\alpha} \subset U^m_{r\alpha} \subset \bar{U}^m_{r\alpha} \subset U^m_{r'\alpha} \subset \bar{U}^m_{r'\alpha} \subset W_{m\alpha} \text{ if } r > r',$$
$$B(U^m_{r\alpha}) = B(U_{mr}) \cap W_{m\alpha}.$$

Hence (3) implies that

(4) order $_p\{B(U^m_{r\alpha}) \mid 0 < r < \sqrt{2}/2m,\ r\text{: rational},\ \alpha \in A_m,\ m=1, 2, \cdots\}$ $\leqq k-1$ for each point $p \in R_k$.

Now, for each m we consider a star-space S_m with the index set A_m and denote by $E_\alpha, \alpha \in A_m$ the unit segments which construct S_m. Let f_m be a mapping of R into S_m defined as follows:

$$f_m(p) = 0 \qquad \text{if } p \notin W_m,$$
$$f_m(p) = \sup \{r \mid p \in U^m_{r\alpha}\} \in E_\alpha \text{ if } p \in W_{m\alpha}.$$
$$(f_m(p) = 0 \text{ if } p \in W_{m\alpha} \text{ and } p \notin U^m_{r\alpha} \text{ for every } r.)$$

Then it is easy to see that f_m is a continuous mapping such that

$$f_m(R - W_m) = 0,$$

$$f_m(F_m)=\sqrt{2}/2m \in E_\alpha \text{ for some } \alpha \in A_m.$$

By use of (2) we can also easily see that

$$f_m(p)=r \in E_\alpha \text{ if and only if } p \in B(U^m_{r\alpha}),$$

i.e. $f_m(p)$ is non-vanishing and rational if and only if $p \in B(U^m_{r\alpha})$ for some $\alpha \in A_m$ and r.

Now we consider the topological product $P=\prod_{m=1}^{\infty} S_m$ and its subset K_n mentioned in the theorem. Let us define a mapping f of R into P by

$$f(p)=\{f_m(p) \mid m=1, 2, \cdots\}.$$

Then it easily follows from the property of f_m and (4) that f is a homeomorphic mapping of R onto a subset of K_n.

In view of the preceding proof we can also assert the following theorem.

Theorem 2. *A metric space R is countable-dimensional, i.e. a countable sum of 0-dimensional spaces if and only if it can be imbedded in the subset K_∞ of a countable product P of star-spaces, where we denote by K_∞ the set of points in P at most finitely many of whose non-vanishing coordinates are rational.*

References

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