64. On a Special Metric Characterizing a Metric Space of $\dim \leqq n$

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Once we have characterized [3] a metric space of covering dimension $\leqq n$ by means of a special metric as follows.

A metric space R has $\dim \leqq n$ *if and only if we can introduce a metric* ρ *in R which satisfies the following condition: For every* $\varepsilon > 0$ *and for every* $n+3$ *points* $x, y_1, \cdots, y_{n+2}$ *in R satisfying*[1]

$$\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon, \quad i=1, \cdots, n+2,$$

there is a pair of indices i, j *such that*

$$\rho(y_i, y_j) < \varepsilon \quad (i \neq j).$$

For separable metric spaces, this theorem was simplified by J. de Groot [2] as follows.

A separable metric space R has $\dim \leqq n$ *if and only if we can introduce a totally bounded metric* ρ *in R which satisfies the following condition:*

For every $n+3$ *points* $x, y_1, \cdots, y_{n+2}$ *in R, there is a triplet of indices,* i, j, k *such that*

$$\rho(y_i, y_j) \leqq \rho(x, y_k) \quad (i \neq j).$$

The first theorem is not so smart though it is valid for every metric space. The problem of generalizing the second theorem, omitting the condition of totally boundedness, to general metric spaces still remains unanswered. However, we can characterize the dimension of a general metric space by a metric satisfying a stronger condition as follows.

Theorem. *A metric space R has* $\dim \leqq n$ *if and only if we can introduce a metric* ρ *into R which satifises the following condition:*

For every $n+3$ *points* $x, y_1, \cdots y_{n+2}$ *in R, there is a pair of indices,* i, j *such that*

$$\rho(y_i, y_j) \leqq \rho(x, y_j) \quad (i \neq j).$$

Proof. The proof of this theorem is never simple.[2] Here we shall only show the proof of sufficiency and the outline of the proof of necessity.

Sufficiency. We shall prove that the following weaker condition is sufficient for R to have $\dim \leqq n$.

We can introduce a metric ρ into R such that for a definite

1) $S_{\varepsilon/2}(x) = \{y \mid \rho(x, y) < \varepsilon/2\}$.

2) The detailed proof will be published in some other place.

number $\delta>0$ and for every $n+3$ points $x, y_1, \cdots, y_{n+2}$ in R with $\rho(x, y_j)<\delta, j=1, \cdots, n+2$, there is a pair of indices i, j such that

$$\rho(y_i, y_j) \leqq \rho(x, y_j) \quad (i \neq j).$$

For $n=0$, the condition for ρ implies that we can introduce a non-Archimedean metric into R. Hence by de Groot's theorem [1] R has dim $\leqq 0$. To prove our assertion by induction we assume its validity and suppose ρ is a metric satisfying the condition for $\delta>0$ and for every $n+4$ points $x, y_1, \cdots y_{n+3}$ in R. Let F be a given closed set of R; then for an arbitrary positive number $\varepsilon<\delta$, we consider the open neighborhood

$$S_\varepsilon(F)=\smile\{S_\varepsilon(y) \mid y \in F\}$$

of F. To assert $\dim R \leqq n+1$ it suffices to show

$$\dim BS_\varepsilon(F) \leqq n$$

where $BS_\varepsilon(F)$ denotes the boundary of $S_\varepsilon(F)$. If we denied the assertion, then by the inductive assumption there would be $n+3$ points $x, y_1, \cdots y_{n+2}$ in $BS_\varepsilon(F)$ such that

$$\rho(x, y_j)<\varepsilon, \ \rho(y_i, y_j)>\rho(x, y_j)$$

for every pair i, j with $i \neq j$. We choose a small neighborhood $U(x)$ of x such that for every point x' of $U(x)$, $\rho(x', y_j)<\varepsilon$ and $\rho(y_i, y_j) >\rho(x', y_j)$ hold. Then there exists a point y_{n+3} of F satisfying $S_\varepsilon(y_{n+3}) \frown U(x) \neq \phi$. Take a point $x' \in S_\varepsilon(y_{n+3}) \frown U(x)$; then

$$\begin{aligned}
&\rho(x', y_j)<\varepsilon<\delta, \quad j=1, \cdots, n+3, \\
&\rho(y_i, y_j)>\rho(x', y_j), \quad i \neq j, \ 1 \leqq i, j \leqq n+2, \\
&\rho(y_i, y_{n+3}) \geqq \varepsilon>\rho(x', y_{n+3}), \quad i=1, \cdots, n+2, \\
&\rho(y_{n+3}, y_j) \geqq \varepsilon>\rho(x', y_j), \quad j=1, \cdots, n+2.
\end{aligned}$$

But this contradicts the property of ρ. Therefore we can conclude that

$$\dim BS_\varepsilon(F) \leqq n$$

and accordingly

$$\dim R \leqq n+1.$$

To carry out the proof of necessity we need the following terminology which is a slight modification of the concept '*rank*' of a collection of sets established in [5] or [6].

Definition. Let $\mathfrak{S}$ be a collection of subsets of R. We call the *Rank* of $\mathfrak{S}$ not greater than n and denote it by $Rank\, \mathfrak{S} \leqq n$ if $\mathfrak{S}$ has the following property.

If $U_1, \cdots, U_l \in \mathfrak{S}$, $\overline{U}_1 \frown \cdots \frown \overline{U}_l \neq \phi$, $U_i \not\subset U_j$ for every pair i, j, with $i \neq j$, then $l \leqq n$.

Necessity. The point of the proof is to define a sequence

(1) $\mathfrak{V}_1>\mathfrak{V}_2^{**}>\mathfrak{V}_2>\mathfrak{V}_3^{**}>\cdots$

of locally finite open coverings such that

(2) $\operatorname{mesh} \mathfrak{V}_m=\sup\{\delta(V) \mid V \in \mathfrak{V}_m\}<1/m$

and a locally finite open covering $\mathfrak{S}'_{m_1\cdots m_p}$ for each sequence $m_1,\cdots,m_p$ of integers with $1\leqq m_1<m_2<\cdots<m_p$ such that

(3) $\mathfrak{S}'_m=\mathfrak{V}_m$

(4) if $2^{-m_1}+\cdots+2^{-m_p}>2^{-l_1}+\cdots+2^{-l_q}$, then $\mathfrak{S}'_{m_1\cdots m_p}>\mathfrak{S}'_{l_1\cdots l_q}$,

(5) $Rank \smile\{\mathfrak{S}'_{m_1\cdots m_p}\mid 1\leqq m_1<m_2<\cdots<m_p\}\leqq n+1$.

Let us decompose R by the decomposition theorem as $R=\bigcup_{i=1}^{n+1}A_i$ for 0-dimensional spaces A_i, $i=1,\cdots,n+1$. Now, we shall define $\mathfrak{V}_1,\mathfrak{V}_2,\cdots,\mathfrak{V}_m$ and $\{\mathfrak{S}'_{m_1\cdots m_p}\mid 1\leqq m_1<\cdots<m_p\leqq m\}$ satisfying the following conditions besides (1), (2), (3), and (4): If we put, for brevity, $\{\mathfrak{S}'_{m_1\cdots m_p}\mid 1\leqq m_1<\cdots<m_p\leqq m\}=\{\mathfrak{S}_1,\cdots,\mathfrak{S}_{k(m)}\}$, then

(6) $U, U'\in\mathfrak{S}_i$ implies either $U\not\subset U'$ or, $U=U'$

(7) $U, U'\in\mathfrak{S}_1\smile\cdots\smile\mathfrak{S}_{k(m)}$ and $U\subsetneqq U'$ imply $\overline{U}\subset U'$,

(8) $Rank\,\mathfrak{S}_1\smile\cdots\smile\mathfrak{S}_{k(m)}\leqq n+1$,

(9) $ord_p B(\mathfrak{S}_1\smile\cdots\smile\mathfrak{S}_{k(m)})\leqq i-1$ for $p\in A_i$,

where for a collection $\mathfrak{S}$ of subsets and a point p of R, $B(\mathfrak{S})$ denotes the collection[3] $\{B(U)\mid U\in\mathfrak{S}\}$ and $ord_p\,\mathfrak{S}$ denotes the greatest number of the members of $\mathfrak{S}$ which contain p.

For $m=1$, we construct a locally finite open covering $\mathfrak{V}'_1=\{V'_\alpha\mid\alpha\in A_1\}$ with $ord\,\overline{\mathfrak{V}}'_1\leqq n+1$, $mesh\,\mathfrak{V}'_1<1$, where for a collection $\mathfrak{V}$ of subsets, $\overline{\mathfrak{V}}$ denotes the collection $\{\overline{V}\mid V\in\mathfrak{V}\}$. Then there is an open covering $\mathfrak{V}''_1=\{V''_\alpha\mid\alpha\in A_1\}$ for which $\overline{V}''_\alpha\subset V'_\alpha$. Then, as we have shown in [4], Lemma 2.1, we can construct open sets V'''_α, $\alpha\in A_1$ such that

$$\overline{V}''_\alpha\subset V'''_\alpha\subset V'_\alpha,$$

$$ord_p\{B(V'''_\alpha)\mid\alpha\in A_1\}\leqq i-1 \quad\text{for}\quad p\in A_i.$$

We choose from $\{V'''_\alpha\mid\alpha\in A_1\}$ the members V'''_α for which $V'''_\alpha\subset V'''_\beta$ ($\beta\in A_1$) implies $V'''_\alpha=V'''_\beta$ and make a collection $\mathfrak{V}_1$ out of them. Then it is easy to see that $\mathfrak{V}_1=\mathfrak{S}'_1$ is the locally finite open covering satisfying all the required conditions.

Now, let us assume that we have already defined $\mathfrak{V}_1,\cdots,\mathfrak{V}_m$ and $\mathfrak{S}_1,\cdots,\mathfrak{S}_{k(m)}$ to define $\mathfrak{V}_{m+1}$ and $\{\mathfrak{S}_{k(m)+1},\cdots,\mathfrak{S}_{k(m+1)}\}=\{\mathfrak{S}'_{m_1\cdots m_p}\mid 1\leqq m_1<\cdots<m_p=m+1\}$. First we construct a locally finite open covering $\mathfrak{V}$ with $mesh\,\mathfrak{V}<1/(m+1)$, $\mathfrak{V}^{**}<\mathfrak{V}_m$ such that

(10) if $U_1,\cdots,U_l\in\mathfrak{S}_1\smile\cdots\smile\mathfrak{S}_{k(m)}$ and $\overline{U}_1\frown\cdots\frown\overline{U}_l=\phi$, then

$$S^3(U_1,\mathfrak{V})\frown\cdots\frown S^3(U_l,\mathfrak{V})=\phi,$$

(11) for each $p\in R$, $S^3(p,\mathfrak{V})$ meets only finitely many members of $\mathfrak{S}_1\smile\cdots\smile\mathfrak{S}_{k(m)}$.

(12) if $U, U'\in\mathfrak{S}_1\smile\cdots\smile\mathfrak{S}_{k(m)}$ and $\overline{U}\subset U'$, then $\overline{S^2(U,\mathfrak{V})}\subset U'$

(13) if $U, U'\in\mathfrak{S}_1\smile\cdots\smile\mathfrak{S}_{k(m)}$ and $\overline{U}\not\supset U'$, then $S^2(U,\mathfrak{V})\not\supset U'$.

3) We often call a collection of subsets a *collection*. $B(U)$ denotes the boundary of U.

Since $\mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m)}$ is locally finite, we can choose such $\mathfrak{V}$. Let $\mathfrak{V}=\{V_\alpha \mid \alpha \in A\}$, then we construct an open covering $\mathfrak{W}'=\{W'_\alpha \mid \alpha \in A\}$ satisfying $\overline{W'_\alpha} \subset V_\alpha$. Since each $S(V_\alpha, \mathfrak{V})$ meets at most finitely many of $U \in \mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m)}$, for each of those U we can define an open set $V_\alpha(U)$ such that

(14) $\overline{W'_\alpha} \subset V_\alpha(U) \subset \overline{V_\alpha(U)} \subset V_\alpha$,

(15) if $U \neq U'$, then either $\overline{V_\alpha(U)} \subset V_\alpha(U')$ or $\overline{V_\alpha(U')} \subset V_\alpha(U)$,

(16) if $U \in \mathfrak{S}'_{m_1 \cdots m_p}$, $U' \in \mathfrak{S}'_{l_1 \cdots l_q}$, $2^{-m_1}+\cdots+2^{-m_p}<2^{-l_1}+\cdots+2^{-l_q}$, then $\overline{V_\alpha(U)} \subset V_\alpha(U')$.

By virtue of (9) we can choose $V_\alpha(U)$ satisfying[4)]

(17) $ord_p\, B(\mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m)} \smile \mathfrak{V}') \leqq i-1$ for $p \in A_i$,

too, where $\mathfrak{V}'=\{V_\alpha(U) \mid \alpha \in A,\ U \in \mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m)}\}$. Suppose $S_{m_1 \cdots m_p}(V) = U$ is a member of $\mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m)}$; then we put

$$S_{m_1 \cdots m_p\, m+1}(V) = \smile\{V_\alpha(U) \mid \alpha \in A,\ S(V_\alpha, \mathfrak{V}) \frown U \neq \phi\},$$
$$\mathfrak{S}'_{m_1 \cdots m_p\, m+1} = \{S_{m_1 \cdots m_p\, m+1}(V) \mid V \in \mathfrak{V}_{m_1}\}.$$

By (11) $\mathfrak{S}'_{m_1 \cdots m_p\, m+1}$ is a locally finite open covering.

(18) We choose only those members of $\mathfrak{S}'_{m_1 \cdots m_p\, m+1}$ which are not contained in any other member and denote the collection of those members also by $\mathfrak{S}'_{m_1 \cdots m_p\, m+1}$. Adding these locally finite open coverings $\mathfrak{S}'_{m_1 \cdots m_p\, m+1}$, $1 \leqq m_1 < \cdots < m_p \leqq m$ to the collection $\sum=\{\mathfrak{S}_1, \cdots, \mathfrak{S}_{k(m)}\}$, we obtain a new collection $\sum'=\{\mathfrak{S}_1, \cdots, \mathfrak{S}_{k(m)}, \mathfrak{S}_{k(m)+1}, \cdots, \mathfrak{S}_{k(m+1)-1}\}$. Then we can see that this collection $\sum'$ of coverings satisfies the conditions (6), (7), (8), and (9). We shall omit the proof in detail, but only notice that the conditions (10), (12), (13), (14), (15), (16), (17), (18) and (4), (6), (7), (8) for $\sum$ are needed for that purpose.

Finally we shall define $\mathfrak{V}_{m+1}=\mathfrak{S}'_{m+1}=\mathfrak{S}_{k(m+1)}$. For the preceding covering $\mathfrak{W}'$ we construct a locally finite open covering $\mathfrak{W}$ such that $\mathfrak{W} < \mathfrak{W}'$,

(19) $$Rank\, \mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m+1)-1} \smile \mathfrak{W} \leqq n+1.$$

Since $\sum'$ satisfies (8) and (9), such a covering $\mathfrak{W}$ can be constructed by a slight modification of the process used in [6], proof of Theorem 2.

Let W be a given member of $\mathfrak{W}$. For every member U of $\mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m+1)-1}$ such that $U \not\supset W$, $U \frown W \neq \phi$, we assign a point $q(W, U) \in W-U$. Then $F(W)=\smile\{q(W, U) \mid U \not\supset W,\ U \frown W \neq \phi,\ U \in \mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m+1)-1}\}$ is a closed set contained in W, because W meets only finitely many members of $\mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m+1)-1}$. Hence by use of (9) for $\sum'$, we can construct an open set $V(W)$ for every $W \in \mathfrak{W}$ such that[5)]

$$F(W) \subset V(W) \subset \overline{V(W)} \subset W,$$
$$ord_p\, \mathfrak{S}_1 \smile \cdots \smile \mathfrak{S}_{k(m+1)-1} \smile \{BV(W) \mid W \in \mathfrak{W}\} \leqq i-1 \text{ for } p \in A_i.$$

4) See [4], Lemma 2.1.

5) See [4], Lemma 2.1.

Put

$$\mathfrak{V}_{m+1}=\mathfrak{S}'_{m+1}=\mathfrak{S}_{k(m+1)}=\{V(W) \mid W\in\mathfrak{W};\ V(W)\subset V(W_0) \text{ and } W_0\in\mathfrak{W} \text{ imply } V(W)=V(W_0)\}.$$

Then it is easy to see from (6), (7) for $\sum'$ and (19) that $\sum''=\{\mathfrak{S}_1,\cdots,\mathfrak{S}_{k(m+1)}\}$ also satisfies (6), (7), (8), and (9). Thus we have defined $\mathfrak{V}_m$, $m=1,2,\cdots$ and $\mathfrak{S}_{m_1\cdots m_p}$, $1\leqq m_1<\cdots<m_p$ which satisfy (1)–(5).

We now introduce a metric ρ into R by use of the coverings $\mathfrak{S}'_{m_1\cdots m_p}$, $1\leqq m_1<\cdots<m_p$ and $\mathfrak{S}'_0=\{R\}$ as follows:

$$\rho(x, y)=\inf\{2^{-m_1}+\cdots+2^{-m_p} \mid y\in S(x, \mathfrak{S}'_{m_1\cdots m_p})\}.$$

The proof that ρ is a metric is a slight modification of the proof of Theorem 5 in [3]. For that proof we need, besides the structure of $S_{m_1\cdots m_p}(V)$, the conditions (1), (2), (3), (4), and (16). Here we shall only prove that the metric ρ satisfies the desired special condition. Let $x, y_1,\cdots, y_{n+2}$ be given n+3 points in R. For every $\varepsilon>0$ we obtain $m_1^j,\cdots, m_{p(j)}^j$, $j=1,\cdots, n+2$ such that

$$\rho(x, y_j)\leqq 2^{-m_1^j}+\cdots+2^{-m_{p(j)}^j}<\rho(x, y_j)+\varepsilon$$

and $U_j\in\mathfrak{S}'_{m_1^j\cdots m_{p(j)}^j}$ such that $x, y_j\in U_j$.

If follows from (5) that there exist U_i and U_j $(i\neq j)$ such that $U_i\subset U_j$. Therefore

$$\rho(y_i, y_j)\leqq 2^{-m_1^j}+\cdots+2^{-m_{p(j)}^j}<\rho(x, y_j)+\varepsilon.$$

We take a pair i, j satisfying

$$\rho(y_i, y_j)<\rho(x, y_j)+\varepsilon_m$$

for a sequence $\{\varepsilon_m\}$ of positive numbers converging to zero. Then we obtain $\rho(y_i, y_j)\leqq\rho(x, y_j)$, proving the necessity. Thus among the conditions (6), (7), (8), (9) for $\mathfrak{S}_1,\cdots,\mathfrak{S}_{k(m)}$ (8) is essential. The other conditions are needed only to continue the inductive argument.

References

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