97. On the Šilov Boundaries of Function Algebras

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Let $C(X)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space X. By a *function algebra* we mean a closed (by supremum norm) subalgebra in $C(X)$ containing constants and separating points of X. For $F\subset X$, let $f|F$ be the restriction of the function f to F and $A|F=\{f|F;f\in A\}$. We easily see that for any closed subset F containing the Šilov boundary ∂A of A (cf. [1], [6]), $A|F$ is closed in $C(F)$. Conversely, it is natural to raise the following question: Let F_0 be a closed subset in X and let $A|F$ be closed in $C(F)$ for any closed subset F containing F_0. Then, does F_0 contain ∂A? The main purpose of this note is to answer the question under certain conditions (Theorems 1 and 2). The proof of Theorem 1 is a modification of that of Glicksberg's theorem (cf. [3]) and we obtain the Glicksberg's theorem as a corollary.

Let A be a function algebra on X. Then there is a unique minimal closed subset E of X such that any continuous function zero on E is in A. This closed subset E is called the *essential set* of A. A is an *essential algebra* if the essential set of A is X (cf. [2]). A function algebra A is said to be an *antisymmetric algebra* (or an *analytic algebra*) if any real-valued function in A is always constant (or any function in A vanishing on a non-empty open set in X is always identically zero) (cf. [4]). An analytic algebra is antisymmetric and an antisymmetric algebra is an essential algebra (cf. [4]).

Our main theorem is the following

Theorem 1. *Let A be an essential algebra and let F_0 be a closed subset in X. If $A|F$ is closed in $C(F)$ for any closed subset F containing F_0, then F_0 contains the Šilov boundary ∂A of A.**)

Proof. We set first $F_1=\{y|y\in X, |f(y)|\leqq \sup_{x\in F_0}|f(x)| \text{ for any } f\in A\}$. Then we see that F_1 is a closed set in X containing F_0. If $F_1=X$, then $\sup_{x\in F_0}|f(x)|=\sup_{x\in F_1}|f(x)|=\sup_{x\in X}|f(x)|$ for any $f\in A$, so $F_0\supset\partial A$. Therefore, in order to prove the theorem we need only to show that $F_1=X$. Suppose the contrary: $X\neq F_1$. We can show first that there is a function $f\in A$ such that $f(x)=1$ on P and $f(x)=0$ on Q for any closed set P and for any closed set Q with $Q\supset F_1$, $P\frown Q=O$. For, let p, q

*) After this paper had been accepted for publication, Prof. I. Glicksberg informed me that this theorem can be also proved by direct use of his theorem [3].

be two distinct points with $p, q \notin F_1$. Since $p \notin F_1$, there is a $g \in A$ such that $1=g(p)>\sup_{x \in F_1}|g(x)|$, so there is a $g_1 \in A$ such that $g_1(p)=1$ and $g_1(x)=0$ for any $x \in F_1$ since $A|F_1 \cup (p)$ is closed in $C(F_1 \cup (p))$. Thus we have a function $g_2 \in A$ such that $g_2(p)=1$ and $g_2(x)=0$ on $F_1 \cup (q)$. Therefore, by the similar method as Glicksberg ([3], p. 159) we obtain a function $f \in A$ such that $f(x)=1$ on P and $f(x)=0$ on Q with $Q \supset F_1$, $P \cap Q = O$.

Now, Katznelson has recently introduced the notion of boundedness of the algebra A at a point in X (cf. [5] and [3]). We have then the following lemmas:

Lemma 1. *Let V_1 and V_2 be open in $X-F_1$, and let A be bounded on each V_i. Then A is bounded on every closed subset F of $V_1 \cup V_2$.*

Lemma 2. *If F is a closed set in $X-F_1$ and A is bounded at each x in F, then there is an open set $V \supset F$ on which A is bounded.*

Lemma 3. *There are at most finitely many x in $X-F_1$ where A is not bounded.*

These lemmas can be proved from the existence of a function $f \in A$ such that $f(x)=1$ on P and $f(x)=0$ on Q with $Q \supset F_1$, $P \cap Q = O$ (by the similar method as [3], Lemmas 1, 2, and 3).

To continue the proof of our theorem, let M be the finite set of all points in $X-F_1$ where A is not bounded. Put $D=F_1 \cup M$. If $D=X$, then $M=X-F_1$ is open and finite, so any point of M is an isolated point. We see easily that an isolated point is bounded, so $D \neq X$. Since A is an essential algebra, there is a continuous function f_0 such that $f_0 \notin A$ and $f_0(x)=0$ for any $x \in D$. Let μ be a complex-valued measure such that $\mu(f_0)=1$ and $\mu(f)=0$ for any $f \in A$. We shall next prove that $\mu(P)=0$ for any closed set P with $P \cap D = O$. Let U_0 be an open set with $X-P \supset \overline{U}_0 \supset U_0 \supset D$. For any positive number ε, we can find an open set $V \supset P$ with $|\mu|(V-P)<\varepsilon$, $V \cap U_0 = O$, where $|\mu|$ denotes the total variation of μ. Set $H=X-U_0$. By Lemma 2 A is bounded on a neighborhood of H; there is a constant c_H for which, whenever K is closed and $K \subset H$, each idempotent in the quotient algebra A/kK has norm $<c_H$, where kK denotes the set $\{f \in A : f(K)=0\}$ (cf. [3], [5]). Since $X-V \supset D$, there is a function $f \in A$ such that $f=1$ on P and $f=0$ on $X-V$. If we set $K=P \cup (H-V)$, the above function f is an idempotent in A/kK. Therefore, there is an $h_1 \in A$ such that $h_1(x)=1$ on P, $h_1(x)=0$ on $H-V$ and $\|h_1\|<c_H$. And we have finally the following function h: $h \in A$, $h(x)=1$ on P, $h(x)=0$ on $X-V$ and $\|h\|<m$ (m is a positive number which is independent of V and so, of ε). In fact, $h=h_1h_0$ is a desired one, if h_0 denotes a fixed function $\in A$ such that $h_0(x)=1$ on P and $h_0(x)=0$ on $\overline{U}_0$.

Thus, $$0=\int_X hd\mu=\int_{X-V} hd\mu+\int_{V-P} hd\mu+\int_P hd\mu$$

and $$\left|\int_{V-P} hd\mu\right|\leqq m|\mu|(V-P)<m\varepsilon,\ \int_{X-V} hd\mu=0,$$

so $$|\mu(P)|=\left|\int_P hd\mu\right|<m\varepsilon.$$

Since ε is arbitrary, $\mu(P)=0$, that is, $\mu(P)=0$ for any closed subset P in $X-D$. Since the function f_0 has the properties that $\mu(f_0)=1$ and $f_0(x)=0$ on D, so $1=\int_X f_0(x)\,d\mu=\int_{X-D} f_0(x)\,d\mu$: But $\int_{X-D} f_0(x)\,d\mu=0$, since $\mu(P)=0$ for any closed subset P in $X-D$. This contradiction shows that $F_0\supset\partial A$.

Remark. (1) In the Theorem 1, the condition of essentiality of A is necessary. Let A_1 be an function algebra on a compact Hausdorff space Y and let $C(Z)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space Z. Let $X=Y\smile Z$ and let Y and Z be both open in X and $Y\frown Z=0$. If we set $A=A_1\oplus C(Z)$ $=\{f\in C(X): f$ equals some $f_1\in A_1$ on Y and equals some $f_2\in C(Z)$ on $Z\}$, A is not an essential algebra. We have easily that $\partial A\supset Z$. Let F_1 be a closed set in X with $F_1\subset Z$, $F_1\neq Z$. If we set $F_0=Y\smile F_1$, we see that for any closed subset $F\supset F_0$, $A|F$ is closed in $C(F)$. But $F_0\not\supset\partial A$ since $F_0\frown Z=F_1$.

(2) We consider the following problem in place of Theorem 1: Let A be an essential algebra and let F_0 be a closed subset in X having the non-empty interior. Let $A|F_0$ be closed in $C(F_0)$. Then, does F_0 contain ∂A? If A is analytic, we can easily prove that $F_0\supset\partial A$ under the above hypothesis. But, if A is essential (or antisymmetric), the conclusion is negative. For example, let X be the set consisting of the unit circle and the origin in the unit disc and let A be the function algebra of the restriction on X of the algebra A_0, where A_0 denotes the set of all continuous function on the unit disc analytic on the open unit disc. If we put the origin as F_0, $F_0\not\supset\partial A$.

By Theorem 1, we have

Theorem 2. *Let A be an arbitrary function algebra and let F_0 be a closed subset in X which is contained in the essential set E of A. If $A|F$ is closed in $C(F)$ for any closed subset F containing F_0, then $\partial A\frown E$ is non-empty and $F_0\supset\partial A\frown E$.*

Proof. By hypothesis, the function algebra $B=A|E$ has the following properties: B is an essential algebra and $B|F$ is closed in $C(F)$ for any closed F in E containing F_0. By Theorem 1, $F_0\supset\partial B$. But $\partial B\supset\partial A\frown E$, since $\partial A\ni x$ if and only if for any neighborhood U of x, there is an $f\in A$ such that $U\supset\{y\,|\,y\in X,\ |f(y)|=\|f\|\}$ (cf. [1]).

Therefore, $F_0 \supset \partial B \supset \partial A \frown E$. We show finally that $\partial A \frown E \neq O$. If $\partial A \frown E = 0$, there is a function $f \in A$ such that $f(x)=0$ on E and $f(x)=1$ on ∂A. If we set $g=1-f$, then $g(x)=0$ on ∂A and $g(x) \not\equiv 0$. This is a contradiction, so $\partial A \frown E \neq O$.

From this theorem we have

Corollary (Glicksberg). Let A be an arbitrary function algebra. If $A|F$ is closed in $C(F)$ for any closed set F in X, then $A=C(X)$.

Proof. If $A \neq C(X)$, then the essential set E has at least two points, say p, q. By hypothesis, $A|F$ is closed in $C(F)$ for any closed $F \ni p$ and $A|F$ is closed in $C(F)$ for any closed $F \ni q$. By Theorem 2, $(p) = \partial A \frown E = (q)$. This contradiction shows that $A=C(X)$.

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