124. On Homotopy Groups $\pi_{2n}(K^n_m, S^n)$

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Let K^n_m be a CW-complex obtained by attaching an $(n+1)$-cell V^{n+1} to the n-sphere S^n by a map of degree $m: S^n \to S^n$ $(n \geqq 3)$, and let $[\alpha, \beta]_r$ denote relative Whitehead product of α and β. Since it is known that $\pi_r(K^n_m, S^n)$ is isomorphic to $\pi_r(S^{n+1})$ if $r<2n$, we have $\pi_{n+1}(K^n_m, S^n) \approx Z[\chi^m_{n+1}]$ where χ^m_{n+1} denotes the characteristic map of V^{n+1} in K^n_m. Now we shall prove the following

Theorem. If n is 3 or 7,

$$\pi_{2n}(K^n_m, S^n) \approx Z_m[\chi^m_{n+1}, \iota_n]_r \oplus \pi_{2n}(S^{n+1}).$$

If either n is even and not 4, 8 or n is 4, 8 and m is even,

$$\pi_{2n}(K^n_m, S^n) \approx Z[\chi^m_{n+1}, \iota_n]_r \oplus \pi_{2n}(S^{n+1}).$$

If n is odd and not 3, 7,

$$\pi_{2n}(K^n_m, S^n) = \chi^m_{n+1*}\pi_{2n}(V^{n+1}, S^n) \smile Z_{2m}[\chi^m_{n+1}, \iota_n]_r$$

and $m[\chi^m_{n+1}, \iota_n]_r = \chi^m_{n+1*}[\bar{\iota}_{n+1}, \iota_n]_r$. Especially we have

Corollary. Let o^n_m denote the order of $[\chi^m_{n+1}, \iota_n]_r$.
Then

If n is 3, 7, o^n_m is m.

If n is odd and not 3, 7, o^n_m is $2m$.

If n is even, o^n_m is infinite.*)

The proof is given in several steps.

Let $\bar{K}^n_m$ be a CW-complex such that $\bar{K}^n_m = K^n_m \smile V^{n+1}$ and $K^n_m \frown V^{n+1} = S^n$. Then we have an exact sequence of the triad $(\bar{K}^n_m, K^n_m, V^{n+1})$,

$$\to \pi_{2n+1}(\bar{K}^n_m, K^n_m, V^{n+1}) \xrightarrow{\partial_+} \pi_{2n}(K^n_m, S^n) \xrightarrow{j_*} \pi_{2n}(\bar{K}^n_m, V^{n+1}) \to .$$

By Theorem of Blaker and Massey Lemma 1 follows from this sequence.

Lemma 1. There exists an exact sequence

$$0 \to \{[\chi^m_{n+1}, \iota_n]_r\} \xrightarrow{i} \pi_{2n}(K_n, S^n) \xrightarrow{p_*} \pi_{2n}(S^{n+1}) \to 0,$$

where $\{\alpha\}$ denotes the cyclic group generated by α and p_* is the induced homomorphism by a map $p: K^n_m \to S^{n+1}$ such that $P(S^n)$ is a base point and $P(K^n_m - S^n)$ is of degree 1.

We are now interested in the kernel of p_*. Let $\boldsymbol{P}$ be the space of paths in K^n_m starting from the base point, whose terminal points are contained in S^n. Since p induces a fibering $\bar{p}: \boldsymbol{P} \to \Omega(S^{n+1})$ with a

*) In [1], James obtained this result in a case that K^n_m is a subcomplex of a total space of an S^n-bundle over S^{n+1}.

$(2n-2)$-connected fibre F, we have an exact sequence:

$$\pi_{2n}(\Omega(S^{n+1}))\xrightarrow{\partial}\pi_{2n-1}(F)\xrightarrow{i_*}\pi_{2n-1}(\boldsymbol{P})\xrightarrow{\bar{p}_*}\pi_{2n-1}(\Omega(S^{n+1}))\to 0.$$

Next consider the commutative diagram

$$\begin{array}{ccc}\pi_{2n}(\Omega(S^{n+1})) & \xrightarrow{\partial} & \pi_{2n-1}(F)\\ \downarrow{\scriptstyle H_1} & & \downarrow{\scriptstyle H_2}\\ H_{2n}(\Omega(S^{n+1})) & \xrightarrow[\tau]{} & H_{2n-1}(F)\end{array}$$

where H_1, H_2 are Hurewicz-homomorphisms and τ is the transgression operator.

By Theorem of Hurewicz H_2 is an isomorphism and H_1 is equivalent to Hopf invariant: $\pi_{2n+1}(S^{n+1})\to Z$. Hence we have

Lemma 2. $P_*^{-1}(0)$ is isomorphic to $H_{2n-1}(F)/\tau H_1\pi_{2n}(\Omega(S^{n+1}))$,
and if n is 3, 7, $H_1\pi_{2n}(\Omega(S^{n+1}))=H_{2n}(\Omega(S^{n+1}))$,
if n is odd and not 3, 7, $H_1\pi_{2n}(\Omega(S^{n+1}))=2H_{2n}(\Omega(S^{n+1}))$,
if n is even, $H_1\pi_{2n}(\Omega(S^{n+1}))=0$.

On the other hand, by considering the fibering:
$\Omega(K_m^n)\to\boldsymbol{P}\to S^n$ we have

Lemma 3. $H_{2n}(\boldsymbol{P}, Z)=0, H_{2n-1}(\boldsymbol{P}, Z)=Z_m$.

Then a part of the exact sequence of the fibering $\bar{p}:\boldsymbol{P}\to\Omega(S^{n+1})$, $H_{2n}(\boldsymbol{P})\to H_{2n}(\Omega(S^{n+1}))\xrightarrow{\tau}H_{2n-1}(F)\to H_{2n-1}(\boldsymbol{P})\to H_{2n-1}(\Omega^{n+1}))$, is transformed into an exact sequence:

$$0\to Z\to H_{2n-1}(F)\to Z_m\to 0.$$

Thus we have the following result by use of the cyclicity of $p_*^{-1}(0)$:

If n is even, $p_*^{-1}(0)$ is isomorphic to Z.

If n is 3, 7, $p_*^{-1}(0)$ is isomorphic to Z_m.

If n is odd and not 3, 7, $p_*^{-1}(0)$ is isomorphic to Z_{2m}.

The corollary is clear by these results. The proof of the Theorem is completed by Lemmas 1, 2, 3, and the following

Lemma 4. If either n is 3, 7 or n is even and not 4, 8 or n is 4, 8 and m is even, there exists a homomorphism $\rho:\pi_{2n}(S^{n+1})\to\pi_{2n}(K_m^n, S^n)$ and it holds that $P_*\circ\rho$ is equal to identity.

Reference

[1] I. M. James: Products on spheres, Mathematica, **6**, 1–13 (1959).