145. A New Algebraical Property of Certain von Neumann Algebras

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1. It has been a subject since F. J. Murray and J. von Neumann [5] that there are two non-hyperfinite, non-isomorphic, continuous finite factors. Recently, J. Schwartz [7] has affirmatively solved the subject by introducing the Property P (Definition 1 below). However, the property P is spatial, and a question still remains to find that a purely algebraical property can serve his need.

In the present note, we shall introduce a purely algebraical property, the property Q (Definition 2), and show that the property Q is sufficient to serve Schwartz' need. Actually, we shall show that the property Q implies the property P in Theorem 1, and that the properties P and Q are equivalent for a group operator algebra in Theorem 2. Besides, we shall show directly that the hyperfinite continuous factor satisfies the property Q in Theorem 5. Furthermore, we shall show that the tensor product of two von Neumann algebras having the property Q satisfies also the property Q in Theorem 6.

2. Let G be a (discrete) group. Let $L^\infty(G)$ be the algebra of all bounded complex-valued functions defined on G. A functional $\int x(g)dg$ on $L^\infty(G)$ will be called a *Banach mean*, cf. [3], when it has the following properties: For $x, y\in L^\infty(G)$ and $g, h\in G$,

1° $\int [\alpha x(g)+\beta y(g)]dg=\alpha\int x(g)dg+\beta\int y(g)dg,$

2° $\int x(g)dg\geqq 0$ if $x(g)\geqq 0$ for all $g\in G,$

3° $\int x(gh)dg=\int x(g)dg,$

4° $\int 1\ dg=1,$

where α and β are complex numbers. According to Day [3], if G has a Banach mean, G will be called an *amenable group*. If $\{T_g \mid g\in G\}$ is a uniformly bounded family of operators on a Hilbert space, then there exists a finite constant K with $|(T_gx\mid y)|\leqq K\|x\|\cdot\|y\|$. Hence $[x\mid y]=\int(T_gx\mid y)dg$ is a bounded conjugate bilinear form on the Hilbert space. Consequently, there exists a unique bounded operator T such that $[x\mid y]=(Tx\mid y)$. We shall call T the *operator Banach mean* on G and write it by $T=\int T_g dg$. A similar construction is already employed by [1] and [6].

It is plain by the definition that the operator Banach mean satisfies the following properties:

a) $\int[\alpha T_g+\beta S_g]dg=\alpha\int T_g dg+\beta\int S_g dg,$

b) $\int T_g dg\geqq 0,$ if $T_g\geqq 0$ for all g,

c) $\int T_g{}^* dg=\left[\int T_g dg\right]^*,$

d) $\int T_{gh}dg=\int T_g dg,$

e) $\int I\,dg=I,$

f) $\int ST_g dg=S\int T_g dg,$

g) $\int T_g S\,dg=\left[\int T_g\,dg\right]S,$

where S is a bounded operator on the Hilbert space. Another, two properties of the operator Banach mean are the following:

h) Let $\mathcal{K}$ be a weakly closed convex set of operators, and suppose $T_g\in\mathcal{K}$ for all $g\in G$, then $\int T_g dg\in\mathcal{K}$.

i) Let T be an operator, and suppose T commutes with T_g for all $g\in G$. Then $\int T_g dg$ commutes with T.

We shall prove h) and i) as follows: Let $T=\int T_g dg$ and suppose in order to proceed by contradiction that $T\bar{\in}\mathcal{K}$. By the well-known separation theorem, there exists a real linear functional φ of operators, continuous in the weak operator topology, and a real constant c, such that $\varphi(T)>c$ and $\varphi(S)\leqq c$ for all $S\in\mathcal{K}$. Since φ is weakly continuous, there exists a set $x_1, x_2,\cdots, x_n,\ y_1, y_2,\cdots, y_n$ of vectors such that $\varphi(S)=\mathrm{Re}\,[\sum_{i=1}^n(Sx_i\,|\,y_i)]$ for each bounded operator S. But then it is clear using the definition and the properties of the operator Banach mean that

$$\varphi(T)=\mathrm{Re}\left[\sum_{i=1}^n\left(\int T_g dg\,x_i\,|\,y_i\right)\right]=\int\mathrm{Re}\sum_{i=1}^n(T_g x_i\,|\,y_i)dg=\int\varphi(T_g)dg\leqq c,$$

which contradiction proves h). By f), g) and the assumption that T_g commutes with T for each $g\in G$,

$$\left[\int T_g\,dg\right]T=\int T_g T\,dg=\int TT_g\,dg=T\left[\int T_g\,dg\right],$$

and so $\int T_g\,dg$ commutes with T, which proves i).

3. Using these properties of the operator Banach mean, we shall investigate the relation between the properties P and Q defined below.

DEFINITION 1. A von Neumann algebra $\mathcal{A}$ has the *property P* if for each linear operator T in the Hilbert space the weakly closed convex hull $\mathcal{K}_T$ of the set $\{UTU^* \mid U \in \mathcal{A}^U\}$, where $\mathcal{A}^U$ it the group of all unitary operators of $\mathcal{A}$, has a non-void intersection with $\mathcal{A}'$.

Clearly the property P depends on the underlying Hilbert space on which $\mathcal{A}$ acts, that is, the property P is spatial.

DEFINITION 2. A von Neumann algebra $\mathcal{A}$ has the *property Q* if there exists an amenable subgroup $\mathcal{G}$ of $\mathcal{A}^U$, which generates $\mathcal{A}$. In this case, G will be called an *amenable generator* of $\mathcal{A}$.

THEOREM 1. *If a von Neuman algebra $\mathcal{A}$ has the property Q, then $\mathcal{A}$ has the property P.*

Proof. Let $\mathcal{G}$ be an amenable generator of $\mathcal{A}$. For any bounded operator T in the Hilbert space on which $\mathcal{A}$ acts, put $\Phi(T) = \int UTU^* dU$ using the operator Banach mean of $\mathcal{G}$. Then $UTU^* \in \mathcal{K}_T$, whence it is clear that $\Phi(T) \in \mathcal{K}_T$ by h). By f), g), and d), $U\Phi(T)U^* = \Phi(T)$ for all $U \in \mathcal{G}$, thus $\Phi(T)$ commutes with $\mathcal{G}$. On the other hand, $\mathcal{G}$ generates $\mathcal{A}$, whence $\Phi(T) \in \mathcal{A}'$. Therefor $\mathcal{K}_T$ meets with $\mathcal{A}'$ by a non-void set.

LEMMA 1. *Let a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathfrak{H}$ have the property Q. Then there exists a mapping σ of the set of all bounded linear operators on $\mathfrak{H}$ into $\mathcal{A}'$ having the following properties*:

(a σ) $\sigma(I)=I$,

(b σ) $\sigma(A'T)=A'\sigma(T)$, for $A' \in \mathcal{A}'$,

(c σ) $\sigma(TA')=\sigma(T)A'$, for $A' \in \mathcal{A}'$,

(d σ) $\sigma(\alpha T+\beta S)=\alpha\sigma(T)+\beta\sigma(S)$,

(e σ) $\sigma(T^*)=\sigma(T)^*$,

(f σ) $\sigma(T) \in \mathcal{K}_T$,

(g σ) $\sigma(T) \in \mathcal{A}'$.

Proof. Let $\mathcal{G}$ be the amenable generator of $\mathcal{A}$. For any operator T, put $\sigma(T)=\int UTU^* dU$, then it is clear that σ satisfies (a σ)$-$(g σ) by the properties of the operator Banach mean.

LEMMA 2. *In the above Lemma 1, suppose furthermore that $\mathcal{A}'$ is finite. Then there exists a linear functional τ defined on all bounded operators having the following properties*:

(a τ) $\tau(I)=1$,

(b τ) $\tau(A'T)=\tau(TA')$, for $A' \in \mathcal{A}'$,

(c τ) $\tau(T) \geqq 0$ if $T \geqq 0$,

(d τ) $\tau(T^*)=\tau(T)^*$.

Proof. Let φ be a normalized trace on $\mathcal{A}'$, cf. [2]. Put $\tau(T) = \varphi[\sigma(T)]$, where $\sigma(T)$ is as in Lemma 1. Then τ has the properties (a τ)$-$(d τ) by Lemma 1 and the property of the trace.

Lemmas 1 and 2 correspond to [7; Lemma 5 and Cor. 6], and implied by the results of Schwartz. However, our proofs are somewhat direct, and may be observed with some interests.

4. Let G be a countable discrete group, and μ be the invariant measure on G which assigns to each point the measure 1. Let $\mathcal{A}(G)$ be the von Neumann algebra generated by the unitary group of all left translations: $V_h f(g)=f(h^{-1}g)$ for $f\in L^2(G)$. Similarly let $\mathcal{A}'(G)$ be the von Neumann algebra generated by all right translations: $U_h f(g)=f(gh)$. By [5; Lemma 5.3.4] $\mathcal{A}(G)$ is a factor if and only if each equivalent class $\{hgh^{-1}\,|\,h\in G\}$, $g\neq 1$, is infinite. Hereafter we shall restrict ourselves that G satisfies always this condition. Then by [5; Lemma 5.3.5] $\mathcal{A}(G)$ and $\mathcal{A}'(G)=\mathcal{A}(G)'$ are continuous and finite. $\mathcal{A}(G)$ will be called the *group operator algebra* of the group G.

THEOREM 2. *If the factor $\mathcal{A}(G)$ has the property P, then $\mathcal{A}(G)$ has the property Q.*

Theorems 1 and 2 show that the property P and the property Q are equivalent for group operator algebras. We shall omit the proof of Theorem 2 since it is contained in the second half of the following theorem by [7; Lemma 7].

THEOREM 3. *$\mathcal{A}(G)$ has the property Q if and only if G is amenable.*

Proof. Suppose that $\mathcal{A}(G)$ has the property Q. For $x\in L^\infty(G)$, let us define an operator T_x on $L^2(G)$ by $T_x f(g)=x(g)f(g)$ for $f\in L^2(G)$. If we put $\int x(g)dg=\tau(T_x)$, where τ is a linear functional of Lemma 2, then it satisfies 1°, 2°, and 4° by Lemma 2. Since

$$\int x(gh)dg=\tau(T_{x(gh)})=\tau(U_h T_x U_h{}^*)=\tau(T_x)=\int x(g)dg,$$

3° is also satisfied, whence G is amenable.

Conversely, suppose that G is amenable. Then $\mathcal{G}=\{V_g\,|\,g\in G\}$ is amenable and clearly generates $\mathcal{A}(G)$, whence $\mathcal{A}(G)$ has the property Q.

Theorems 1 and 3 are sufficient to reproduce the theorem of Schwartz [7] combining with the known theorems on amenable groups. If Π is the group of all those permutations of countably many objects which leave all but a finite set unchanged, then Π is amenable by [3; p. 516, (F)], whence $\mathcal{A}(\Pi)$ has the property Q by Theorem 3. If Φ is the free group on two generators, then Φ is not amenable by [3; p. 516, (G)], whence $\mathcal{A}(\Phi)$ has not the property Q. On the other hand, by [5; Lemma 6.2.2], $\mathcal{A}(\Phi)$ has not Property Γ of [5; Def. 6.1.1] and $\mathcal{A}(\Pi\times\Phi)=\mathcal{A}(\Pi)\otimes\mathcal{A}(\Phi)$ satisfies Property Γ by a theorem of Misonou [4] and [5; Lemma 6.1.2], whence $\mathcal{A}(\Phi)$ and $\mathcal{A}(\Pi\times\Phi)$ are not isomorphic. Since $\mathcal{A}(\Pi\times\Phi)$ is not hyperfinite *$\mathcal{A}(\Phi)$ and $\mathcal{A}(\Pi\times\Phi)$ is a pair of non-isomorphic, non-hyperfinite,* *continuous finite factors* as shown in [7].

5. In the remainder, we shall give a few algebraic facts on the property Q.

THEOREM 4. *The property Q is purely algebraical.*

This is clear by the fact that a group isomorphic to an amenable group is amenable.

LEMMA 3. *Let $\mathcal{A}_n$ have the property Q and $\mathcal{G}_n$ be an amenable generator of $\mathcal{A}_n$ for each n. Let $\mathcal{A}_n$ are monotonically increasing and generate a von Neumann algebra $\mathcal{A}$, and let $\mathcal{G}_n \subset \mathcal{G}_m$ for $n < m$. Then $\mathcal{A}$ has the property Q.*

Poof. Put $\mathcal{G} = \bigcup_n \mathcal{G}_n$. Then $\mathcal{G}$ is an amenable generator of $\mathcal{A}$ by [3; p. 516, (F)] and [1; Theorem 2].

THEOREM 5. *The hyperfinite factor has the property Q.*

This is a consequence of Theorem 3 and the fact that Π is amenable since $A(\Pi)$ is hyperfinite. However, we shall give here a direct proof basing on the idea of Schwartz [7; Lemma 2]. By Lemma 3, it is sufficient to show that a discrete finite factor $\mathcal{B}$ has the property Q. This is obvious since $\mathcal{B}$ has a finite amenable generator.

THEOREM 6. *If von Neumann algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ have the property Q, then the tensor product $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ has the property Q too.*

Proof. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ are amenable generators of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. If $\mathcal{G} = \{U_1 \otimes U_2 \mid U_1 \in G_1, U_2 \in \mathcal{G}_2\}$, then $\mathcal{G}$ generates $\mathcal{A}$ by [2; Chap. I, §2, Prop. 6]. Since $\mathcal{G}$ is isomorphic to $\mathcal{G}_1 \times \mathcal{G}_2$, $\mathcal{G}$ is amenable by [3; p. 517, (F'')]. Therefore $\mathcal{G}$ is an amenable generator of $\mathcal{A}$.

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