42. On the Lebesgue Constants for Quasi-Hausdorff Methods of Summability. I

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§ 1. The quasi-Hausdorff transformation (H^*, ψ) is defined as transforming the sequence $\{s_\nu\}$ into the sequence $\{h_n^*\}$ by means of the equation

$$h_n^* = \sum_{\nu=n}^{\infty} \binom{\nu}{n} s_\nu \int_0^1 r^{n+1}(1-r)^{\nu-n} d\psi(r),$$

where the weight function $\psi(r)$ is of bounded variation in the interval $0 \leq r \leq 1$. This transformation is regular if and only if

$$\psi(1) - \psi(+0) = 1.$$

We may assume, in the following, that

$$\psi(1) = 1, \quad \psi(+0) = 0.$$

Corresponding to any fixed number r with $0 < r \leq 1$, if we put $\psi(x) = e_r(x)$, where

$$e_r(x) = \begin{cases} 0 & \text{for } 0 \leq x < r \\ 1 & \text{for } r \leq x \leq 1, \end{cases}$$

then the quasi-Hausdorff transformation reduces to the circle transformation (γ, r).

The Lebesgue constant of order n for the method (H^*, ψ) is then defined to be

(1.1) $$L^*(n; \psi) = \\ = \frac{2}{\pi} \int_0^\pi dt \left| \sum_{\nu=n}^{\infty} \binom{\nu}{n} \frac{\sin(\nu + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \int_0^1 r^{n+1}(1-r)^{\nu-n} d\psi(r) \right|.$$

As is well known, if $L^*(n; \psi) \to \infty$ as $n \to \infty$, then there is a continuous function whose Fourier series is not summable (H^*, ψ) for at least one point.

The Lebesgue constants for the method (γ, r) were studied by L. Lorch [4] and by the author [2]. On the other hand, first A. E. Livingston [3] and recently L. Lorch and D. J. Newman [4] studied the Lebesgue constants for the regular Hausdorff methods of summability in detail. For the definition and the properties of the Hausdorff methods, see, e.g., G. H. Hardy [1]. We shall study, in this note, the Lebesgue constants for the quasi-Hausdorff methods of summability.

§ 2. From (1.1), we get

$$(2.1)\qquad L^*(n;\psi)=\\ =\frac{2}{\pi}\int_0^{\pi/2}\frac{du}{\sin u}\left|\int_0^1 r^{n+1}\mathcal{I}\left\{\frac{e^{i(2n+1)u}}{(1-e^{2iu}+re^{2iu})^{n+1}}\right\}d\psi(r)\right|.$$

Here we put

$$\frac{1}{1-e^{2iu}+re^{2iu}}=p(u,r)e^{iq(u,r)},$$

then

$$1-\cos 2u+r\cos 2u=\frac{1}{p(u,r)}\cos q(u,r)$$

$$\sin 2u-r\sin 2u=\frac{1}{p(u,r)}\sin q(u,r)$$

$$(2.2)\qquad \{rp(u,r)\}^2=\frac{r^2}{r^2+4(1-r)\sin^2 u}$$

$$0\leq rp(u,r)\leq 1,$$

where $rp(u,r)=1$ if, and only if, $u=0$ or $r=1$.

Then, from (2.1), we obtain

$$L^*(n;\psi)=\\ =\frac{2}{\pi}\int_0^{\pi/2}du\left|\int_0^{1-0}\frac{1}{\sin u}r^{n+1}p^{n+1}(u,r)\sin\{(n+1)q(u,r)+(2n+1)u\}d\psi(r)+\right.\\ \left.+\frac{\sin(2n+1)u}{\sin u}[\psi(1)-\psi(1-0)]\right|.$$

Since

$$\left|(e^{iu}-1)\left(\frac{1}{\sin u}-\frac{1}{u}\right)\mathcal{I}\left\{\frac{r^{n+1}e^{i(2n+1)u}}{(1-e^{2iu}+re^{2iu})^{n+1}}\right\}\right|\leq M<\infty$$

for $0<u\leq\frac{\pi}{2}$, $0\leq r\leq 1$ and tends to zero as $n\to\infty$ except on the line $r=1$, we have

$$(2.3)\qquad L^*(n;\psi)=\\ =\frac{2}{\pi}\int_0^{\pi/2}du\left|\int_0^{1-0}\frac{1}{u}r^{n+1}p^{n+1}(u,r)\sin\{(n+1)q(u,r)+2(n+1)u\}d\psi(r)+\right.\\ \left.+\frac{\sin(2n+1)u}{\sin u}[\psi(1)-\psi(1-0)]\right|+o(1).$$

as $n\to\infty$.

§ 3. From the previous paper [2], it is easily seen that the estimate, for small u,

$$(3.1)\qquad q(u,r)=2\frac{1-r}{r}u+O(u^3)$$

for *fixed* r holds *uniformly* in r for $0<\delta\leq r\leq 1$ with any fixed δ. Here we shall estimate

$$(3.2)\qquad L^*_\delta(n;\psi)=\\ =\frac{2}{\pi}\int_0^{\pi}dt\left|\sum_{\nu=n}^{\infty}\binom{\nu}{n}\frac{\sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t}\int_\delta^1 r^{n+1}(1-r)^{\nu-n}d\psi(r)\right|$$

$$=\frac{2}{\pi}\int_0^{\pi/2}du\left|\int_\delta^{1-0}\frac{1}{u}r^{n+1}p^{n+1}(u,r)\sin\{(n+1)q(u,r)+2(n+1)u\}d\psi(r)+\right.$$
$$\left.+\frac{\sin(2n+1)u}{\sin u}[\psi(1)-\psi(1-0)]\right|+o(1).$$

Let

$$I_n=\frac{2}{\pi}\int_0^{\pi/2}du\left|\int_\delta^{1-0}\frac{1}{u}r^{n+1}p^{n+1}(u,r)\sin\{(n+1)q(u,r)+2(n+1)u\}d\psi(r)+\right.$$
$$\left.+\frac{\sin(2n+1)u}{\sin u}[\psi(1)-\psi(1-0)]\right|-$$
$$-\frac{2}{\pi}\int_0^{\pi/2}du\left|\int_\delta^{1-0}\frac{1}{u}r^{n+1}p^{n+1}(u,r)\sin 2(n+1)\frac{u}{r}d\psi(r)+\right.$$
$$\left.+\frac{\sin(2n+1)u}{\sin u}[\psi(1)-\psi(1-0)]\right|.$$

From the previous paper [2], it is easily seen that if $1<m<e$, then

$$r^2p^2(u,r)<m^{-\frac{8(1-r)}{\pi^2r^2}u^2} \tag{3.3}$$

holds for r in the interval $\delta\le r\le 1$ and for sufficiently small $u\ge 0$.

Now we shall take $\sigma>0$ such that, in the interval $0\le u\le\sigma$, (3.1) and (3.3) hold simultaneously. Then

$$|I_n|\le O(n+1)\int_0^\sigma du\int_\delta^{1-0}r^{n+1}p^{n+1}(u,r)u^2|d\psi(r)|+$$
$$+\frac{4}{\pi\sigma}\int_\sigma^{\pi/2}du\int_\delta^{1-0}r^{n+1}p^{n+1}(u,r)|d\psi(r)|.$$

From the Lebesgue principle of dominated convergence, we have $\lim_{n\to\infty} I_n=0$. Hence

$$L_\delta^*(n;\psi)=\frac{2}{\pi}\int_0^{\pi/2}du\left|\int_\delta^{1-0}\frac{1}{u}r^{n+1}p^{n+1}(u,r)\sin 2(n+1)\frac{u}{r}d\psi(r)+\right. \tag{3.4}$$
$$\left.+\frac{\sin(2n+1)u}{\sin u}[\psi(1)-\psi(1-0)]\right|+o(1).$$

§ 4. Here we shall prove the following

Theorem 1. *If the weight function $\psi(r)$ is a step-function which is continuous at the origin, then*

$$L^*(n;\psi)=C^*(\psi)\log n+o(\log n)\qquad as\ n\to\infty, \tag{4.1}$$

where

$$C^*(\psi)=\frac{2}{\pi^2}|\psi(1)-\psi(1-0)|+\frac{1}{\pi}\mathcal{M}\left\{\left|\sum_k[\psi(\xi_k+0)-\psi(\xi_k-0)]\cdot\sin\frac{u}{\xi_k}\right|\right\}. \tag{4.2}$$

Here ξ_k is the k-th discontinuity (jump) of $\psi(r)$ and the summation extends over all such (possibly countably infinite) values, $\mathcal{M}\{f(u)\}$

represents the mean value of the almost periodic function $f(u)$.

This theorem corresponds to the following one of L. Lorch and D. J. Newman [5]. They studied the Lebesgue constants $L(n;\psi)$ for the general regular Hausdorff methods of summability in detail:

Theorem 2. *Under the same condition on* $\psi(r)$ *as in Theorem* 1, *we obtain*

$$L(n;\psi)=C(\psi)\log n+o(\log n) \quad \text{as } n\to\infty, \tag{4.3}$$

where

$$C(\psi)=\frac{2}{\pi^2}|\psi(1)-\psi(1-0)|+\frac{1}{\pi}\mathcal{M}\{|\sum_k[\psi(\xi_k+0)-\psi(\xi_k-0)]\cdot \sin\xi_k u|\}. \tag{4.4}$$

We see easily symmetric relation between $C^*(\psi)$ and $C(\psi)$.

(References are listed at the end of the next article, p. 195.)