39. On Metrizability of M-Spaces

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§ 1. **Introduction.** Let X be a topological space. An open covering $\mathfrak{U}$ of X is said to be a *star-refinement* of another open covering $\mathfrak{V}$ of X if the covering $\{St(U, \mathfrak{U}) | U \in \mathfrak{U}\}$ is a refinement of $\mathfrak{V}$ where $St(A, \mathfrak{U})$ means the union of the sets U of $\mathfrak{U}$ such that $A \cap U \neq \phi$. A sequence $\{\mathfrak{U}_n | n=1, 2, \cdots\}$ of open coverings of X is said to be *normal* if $\mathfrak{U}_{n+1}$ is a star-refinement of $\mathfrak{U}_n$ for $n=1, 2, \cdots$.

We shall say that a topological space X is an *M-space* if there exists a normal sequence $\{\mathfrak{U}_n | n=1, 2, \cdots\}$ of open coverings of X satisfying the condition $(*)$ below:

$(*)$ If a family $\mathfrak{A}$ consisting of a countable number of subsets of X has the finite intersection property and contains as a member a subset of $St(x_0, \mathfrak{U}_n)$ for every n and for some fixed point x_0 of X, then $\bigcap\{\bar{A} | A \in \mathfrak{A}\} \neq \phi$.

Metrizable spaces and countably compact spaces are clearly M-spaces.

The notion of M-spaces was introduced and discussed in [5].

Theorem 1. *Let X be a topological space. In order that X be metrizable it is necessary and sufficient that X be a paracompact Hausdorff M-space and that the product space $X \times X$ be perfectly normal.*

More precisely, we shall obtain the theorem below:

Theorem 1′. *Let X be a topological space. In order that X be metrizable it is necessary and sufficient that X be a paracompact Hausdorff M-space and that the diagonal Δ of the product space $X \times X$ be a G_δ-set in $X \times X$.*

It is easily seen that Theorem 1 is deduced from Theorem 1′. Therefore, we have only to prove Theorem 1′; this will be done in §2.

A completely regular space X is said to be *absolute* G_δ if it is a G_δ-set in every extension of it, that is, if X is a dense subset of a completely regular space Y, then X is a G_δ-set in Y.

It is well known that a metrizable space is absolute G_δ if and only if it is completely metrizable (cf. [1]).

Z. Frolik has proved that a paracompact normal space which is absolute G_δ is an M-space. More generally, K. Morita ([7], [8]) has proved that a paracompact normal space which is G_δ in a countably compact space is an M-space.

Combining these results and Theorem 1′ we have the following theorem.

Theorem 2. *Let X be a topological space. In order that X be completely metrizable it is necessary and sufficient that X be a paracompact Hausdorff space which is absolute G_δ and that the diagonal of the product space $X\times X$ be a G_δ-set in $X\times X$.*

Let K be a CW-complex in the sense of J. H. C. Whitehead [11]. Then K is a paracompact Hausdorff space; this was proved in [4]. And the product space $K\times K$ is perfectly normal (cf. [6]). But K is not an M-space in general (cf. [6]). If we apply Theorem 1 to K, we can establish the result below:

Theorem 3. *Let K be a CW-complex. Then the following conditions on K are equivalent.*

(a) *K is abolute G_δ.*

(b) *K is locally compact.*

(c) *K is an M-space.*

(d) *K is metrizable.*

(e) *K is completely metrizable.*

Finally, in §4 we will discuss some remarks to the sufficient conditions in Theorems 1 and 1′.

§2. **Proof of Theorem 1′.** Since X is metrizable, $X\times X$ is metrizable, we have only to prove the sufficiency of the conditions.

Let $\varDelta$ be the diagonal of $X\times X$, that is, $\varDelta=\{(x, x)\mid x\in X\}$. Since $\varDelta$ is a G_δ-set in $X\times X$, there exists a countable collection $\{G_n\mid n=1, 2, \cdots\}$ of open sets of $X\times X$ such as $\bigcap_{n=1}^{\infty} G_n=\varDelta$.

Let $\{\mathfrak{U}_n\mid n=1, 2, \cdots\}$ be a normal sequence of open coverings of X which satisfies the condition $(*)$ of M-space. Since each G_n contains $\varDelta$, for arbitrary point x of X we can choose a neighborhood $U_n(x)$ of x such that $U_n(x)\times U_n(x)\subset G_n$ for $n=1, 2, \cdots$. By the paracompactness of X there exists a normal sequence $\{\mathfrak{V}_n\mid n=1, 2, \cdots\}$ of locally finite open coverings of X such that each $\mathfrak{V}_n$ is a refinement of the covering $\{U_n(x)\mid x\in X\}$ of X.

Let us put $\mathfrak{W}_n=\{U\cap V\mid U\in\mathfrak{U}_n,\ V\in\mathfrak{V}_n\}$ for $n=1, 2, \cdots$. Then the sequence $\{\mathfrak{W}_n\mid n=1, 2, \cdots\}$ has the following properties (1) and (2).

(1) $\{\mathfrak{W}_n\mid n=1, 2, \cdots\}$ satisfies the condition $(*)$ of M-space.

(2) $\bigcup\{W\times W\mid W\in\mathfrak{W}_n\}\subset G_n$ for $n=1, 2, \cdots$.

Indeed, (1) follows from the fact that $St(x, \mathfrak{W}_n)\subset St(x, \mathfrak{U}_n)$ is true for each n and each point x of X, and (2) is an immediate consequence of the constructions of $\mathfrak{V}_n$ and $\mathfrak{W}_n (n=1, 2, \cdots)$.

Now we shall show that $\{St(x, \mathfrak{W}_n)\mid n=1, 2, \cdots\}$ is a neighborhood basis at x for every point x of X.

Suppose that it is contrary. Then there exist a point x of X and a neighborhood N of x such that $St(x, \mathfrak{W}_n)-N$ is not empty for

$n=1, 2, \cdots$. Let x_n be a point of X such that

(3) $$St(x, \mathfrak{W}_n)-N \ni x_n \quad \text{for} \quad n=1, 2, \cdots,$$

and let us put $A_n=\{x_k \mid k>n\}$ for each n.

Since $\{\mathfrak{W}_n \mid n=1, 2, \cdots\}$ is a normal sequence, we can choose an element W_n of $\mathfrak{W}_n$ such that

$$St(x, \mathfrak{W}_{n+1}) \subset W_n \quad \text{for } n=1, 2, \cdots.$$

Hence, we have $A_{n+1} \subset W_{n+1}$ and $\bar{W}_{n+1} \subset W_n$. Therefore, we have

(4) $$\bar{A}_{n+1} \subset W_n \quad \text{for } n=1, 2, \cdots.$$

Using (2) and (4) it is concluded that

$$\{x\} \times (\bigcap_{n=1}^{\infty} \bar{A}_n) = \bigcap_{n=1}^{\infty} (\{x\} \times \bar{A}_n) \subset \bigcap_{n=1}^{\infty} (W_n \times W_n) \subset \bigcap_{n=1}^{\infty} G_n = \Delta.$$

Therefore, we have

(5) $$\{x\} \times (\bigcap_{n=1}^{\infty} \bar{A}_n) \subset \Delta.$$

By the condition (*) of M-space we have

(6) $$\bigcap_{n=1}^{\infty} A_n \neq \phi.$$

Consequently, we have to obtain, by (5) and (6),

(7) $$\bigcap_{n=1}^{\infty} A_n = \{x\}.$$

On the other hand, from (3) we have to get the conclusion that $\bar{A}_n \not\ni x$ for each n. But this contradicts to (7).

Hence, it is shown that $\{St(x, \mathfrak{W}_n) \mid n=1, 2, \cdots\}$ is a neighborhood basis at x for any point x of X. Therefore, X is metrizable (cf. [9]).

§ 3. **Proof of Theorem 3.** We have that (a) implies (c) for paracompact normal spaces (cf. [3], [7], and [8]).

According to Theorem 1 we can obtain that (c) implies (d).

Now we shall prove that (d) implies (b). For this purpose assume (d). Then it is sufficient to show that K is a locally finite complex meaning that each point x of K is an inner point of some finite subcomplex of K (cf. [11]). Contrary to this conclusion, suppose that there is a point x_0 of K at which K is not locally finite. Then we can choose a sequence $\{x_n \mid n=1, 2, \cdots\}$ of points of K and a sequence $\{K(e_n)^{*)} \mid n=1, 2, \cdots\}$ in K such that

(1) $\rho(x_0, x_n)<\frac{1}{n}$, where ρ is the given metric function on K,

and

(2) $$x_n \notin K(e_1) \cup \cdots \cup K(e_{n-1}), \; x_n \in K(e_n)$$

are satisfied for each n. Indeed, let $\{x_1, \cdots, x_{n-1}\}$ and $\{K(e_1), \cdots, K(e_{n-1})\}$ satisfying (1) and (2) be chosen. Then, by the assumption of x_0, the set $U\left(x_0, \frac{1}{n}\right)=\left\{x \mid x \in K, \rho(x_0, x)<\frac{1}{n}\right\}$ intersects infinitely

*) We shall use $K(e_n)$ to stand for the intersection of all the subcomplexes of K, which contain e_n (cf. [11]).

many closed cells of K. Let $\bar{e}_n$ be one of the closed cells which intersects $U\left(x_0, \frac{1}{n}\right)$ and which is not contained in $K(e_1)\smile\cdots\smile K(e_{n-1})$, and let x_n be a point of $K(e_n)-K(e_1)\smile\cdots\smile K(e_{n-1})$. It is easily seen that $\{x_1, \cdots, x_n\}$ and $\{K(e_1), \cdots, K(e_n)\}$ satisfy (1) and (2). Hence, by induction, we can obtain $\{x_n | n=1, 2, \cdots\}$ and $\{K(e_n) | n=1, 2, \cdots\}$ which are desirable.

Since K is a closure finite complex and has the weak topology, $\{x_n | n=1, 2, \cdots\}$ is a closed set of K. On the other hand, by (1) x_0 is the limit point of $\{x_n | n=1, 2, \cdots\}$. This contradiction shows that K is locally finite (cf. [10]).

It is well known that (b) implies (a) for more general spaces.

It is clear that (e) implies (d).

Finally, we obtain that (a) implies (e) by Theorem 2.

Thus, the proof of Theorem 3 is completed.

§ 4. Remarks to Theorem 1 and 1′.

1. The following example shows that Theorem 1′ is not true without the assumption that Δ is a G_δ-set in $X\times X$ even though X is perfectly normal.

Let X be the union of the top and bottom edges of the unit square topologized by dictionary ordering. Then X is a non-metrizable, compact Hausdorff space which is separable and has a countable base at each point (cf. [2], Example 6.3). Moreover, it is easily seen that X is perfectly normal.

2. Let X be a subspace of βN which is a union of integers N and one point of $\beta N-N$. Then X is a non-metrizable, σ-compact Hausdorff space (therefore, paracompact) space (cf. [2], Example 6.2) and $X\times X$ is perfectly normal (cf. [6], Corollary to Theorem 1).

This example shows that Theorem 1 is not true even if X is a countable union of closed subsets each of which is an M-space instead of the assumption that X is an M-space.

References

[1] E. Čech: On bicompact spaces. Ann. of Math. (2), **38**, 823-844 (1937).
[2] H. H. Corson and E. Michael: Metrizability of countable union, to appear.
[3] Z. Frolik: On the topological product of paracompact spaces. Bull. de l'academie Polonaise de Sciences, **8**, 747-750 (1960).
[4] K. Morita: On spaces having the weak topology with respect to a closed covering I, II. Proc. Japan Acad., **29**, 537-543 (1953); **30**, 711-717 (1954).
[5] ——: On the products of a normal spaces with a metric space. Proc. Japan Acad., **39**, 148-150 (1963).
[6] ——: On the product of paracompact spaces. Proc. Japan Acad., **39**, 559-563 (1963).
[7] ——: Products of normal spaces with metric spaces, to appear.
[8] ——: Products of normal spaces with metric spaces II, to appear.
[9] J. W. Tukey: Convergence and uniformity in topology. Princeton (1940).
[10] J. H. C. Whitehead: Simplicial spaces, nuclei and m-groups. Proc. London Math. Soc., **45**, 243-327 (1938).
[11] ——: Combinatorial homotopy I. Bull. Amer. Math. Soc., **55**, 213-245 (1949).