60. On the Covering Dimension of Product Spaces

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Let X and Y be normal spaces. As for the covering dimension of the product space $X\times Y$ we have known several cases for which the following relation

(A) $$\dim(X\times Y)\leqq \dim X+\dim Y$$

holds.

Especially when Y is a separable metrizable space, (A) has been proved in each of the following cases.

(a) X is metrizable ([2]).

(b) X is countably paracompact and normal, and Y is locally compact ([2]).

In the present paper we shall prove (A) under the conditions that Y is separable metrizable and $X\times Y$ is countably paracompact and normal.

Recently E. Michael [1] has given a non-normal space $X\times Y$ which is a product space of a hereditarily paracompact normal space X with a separable metric space Y. This space $X\times Y$ is not 0-dimensional, nevertheless X and Y are 0-dimensional; thus (A) does not hold.

Accordingly the normality of $X\times Y$ is indispensable.

The idea of the proof for our theorem is based on the "basic coverings" introduced by K. Morita ([3]).

1. Henceforth Y always means a separable metrizable space.

Lemma 1. *Suppose that* $\dim Y=n$ *and let* s *be an arbitrary positive integer: then there are locally finite countable coverings*

$$\mathfrak{V}_i^{(l)}=\{V_{i\alpha}^{(l)}\mid \alpha=1,2,\cdots\}\ (1\leqq l\leqq s;\ i=1,2,\cdots)$$

satisfying the following conditions (i) *and* (ii).

(i) $\bigcup_i \mathfrak{V}_i^{(l)}$ *is an open basis of* Y *for any* $l\,(1\leqq l\leqq s)$.

(ii) *The order of the family* $\{\mathfrak{B}V_{i\alpha}^{(l)}\mid i,\alpha=1,2,\cdots;1\leqq l\leqq s\}$ *is at most* n. (Here $\mathfrak{B}V_{i\alpha}^{(l)}$ means $\overline{V_{i\alpha}^{(l)}}-V_{i\alpha}^{(l)}$.)

Proof. The existence of $\mathfrak{V}_i^{(l)}$ satisfying (i) is well known (e.g. [3]), and these may be considered as countable coverings for any i and l, according to separability of Y. Moreover, the existence of such $\mathfrak{V}_i^{(l)}$ that satisfy (ii) is assured by the shrinkability of the covering $\mathfrak{V}_i^{(l)}$ and [4].

Put $$W^{(l)}(\alpha_1,\alpha_2,\cdots,\alpha_i)=V_{1\alpha_1}^{(l)}\cap V_{2\alpha_2}^{(l)}\cap\cdots\cap V_{i\alpha_i}^{(l)}.$$

Lemma 2. *Let* $(\alpha_1,\cdots,\alpha_i)$, $(\beta_1,\cdots,\beta_j),\cdots,(\lambda_1,\cdots,\lambda_h)$ *be* $n+1$ *sets, each of which is a finite ordered set of positive integers. Then* $\mathfrak{B}W^{(l_1)}(\alpha_1,\cdots,\alpha_i)\cap\mathfrak{B}W^{(l_2)}(\beta_1,\cdots,\beta_j)\cap\cdots\cap\mathfrak{B}W^{(l_{n+1})}(\lambda_1,\cdots,\lambda_h)=\phi$ for $1\leqq l_1<l_2<\cdots<l_{n+1}\leqq s$.

Proof. According to Lemma 1 (ii) we have

$$(\mathfrak{B}V_{1\alpha_1}^{(l_1)}\cup\mathfrak{B}V_{2\alpha_2}^{(l_1)}\cup\cdots\cup\mathfrak{B}V_{i\alpha_i}^{(l_1)})\cap(\mathfrak{B}V_{1\beta_1}^{(l_2)}\cup\cdots\cup\mathfrak{B}V_{j\beta_j}^{(l_2)})\cap\cdots\cap(\mathfrak{B}V_{1\lambda_1}^{(l_{n+1})}\cup\cdots\cup\mathfrak{B}V_{h\lambda_h}^{(l_{n+1})})=\phi.$$

But $\mathfrak{B}W^{(l)}(\alpha_1,\cdots,\alpha_i)\subset(\mathfrak{B}V_{1\alpha_1}^{(l)}\cup\cdots\cup\mathfrak{B}V_{i\alpha_i}^{(l)})$, thus the lemma is proved.

Theorem. *If a product space* $X\times Y$ *of a space* X *with a separable metrizable space* Y *is countably paracompact and normal, then*

$$\dim(X\times Y)\leqq\dim X+\dim Y.$$

Proof. Suppose $\dim X=m$ and $\dim Y=n$, and put $s=m+n+1$.

Let $F^{(l)}$ and $G^{(l)}$ be arbitrarily given closed sets and open sets respectively such that $F^{(l)}\subset G^{(l)}$ $(1\leqq l\leqq s)$.

There are open sets $L^{(l)}$ and $M^{(l)}(1\leqq l\leqq s)$ such that

$$F^{(l)}\subset M^{(l)}\subset\overline{M^{(l)}}\subset L^{(l)}\subset\overline{L^{(l)}}\subset G^{(l)}.$$

We put

$$N_1^{(l)}=X\times Y-\overline{M^{(l)}},\ N_2^{(l)}=L^{(l)}.$$

Then $\mathfrak{N}^{(l)}=\{N_1^{(l)},N_2^{(l)}\}$ is an open covering of $X\times Y$ for any l.

We put

(1) $G^{(l)}(\alpha_1,\cdots,\alpha_i;k)=\mathrm{Int}\{x\,|\,x\times W^{(l)}(\alpha_1,\cdots,\alpha_i)\subset N_k^{(l)}\}$ $(k=1,2)$.

(Here $\mathrm{Int}\,A$ means the interior of the subset A.)

Then $G^{(l)}(\alpha_1,\cdots,\alpha_i;k)\times W^{(l)}(\alpha_1,\cdots,\alpha_i)\subset N_k^{(l)}$. By (1) we get immediately $G^{(l)}(\alpha_1,\cdots,\alpha_i;k)\subset G^{(l)}(\alpha_1,\cdots,\alpha_i,\alpha_{i+1};k)$.

Put $G^{(l)}(\alpha_1,\cdots,\alpha_i)=G^{(l)}(\alpha_1,\cdots,\alpha_i;1)\cup G^{(l)}(\alpha_1,\cdots,\alpha_i;2)$.

Then

$$G^{(l)}(\alpha_1,\cdots,\alpha_i)\subset G^{(l)}(\alpha_1,\cdots,\alpha_i,\alpha_{i+1}).$$

Consequently $\{G^{(l)}(\alpha_1,\cdots,\alpha_i)\times W^{(l)}(\alpha_1,\cdots,\alpha_i)\,|\,\alpha_1,\cdots,\alpha_i;i\}$ is a basic covering ([3]) for each l.

Now, since $X\times Y$ is countably paracompact and normal, we get a special refinement ([3]). That is to say, there exists a family $\{F^{(l)}(\alpha_1,\cdots,\alpha_i)\,|\,\alpha_1,\cdots,\alpha_i;i\}$ of closed sets in X such that

$$F^{(l)}(\alpha_1,\cdots,\alpha_i)\subset G^{(l)}(\alpha_1,\cdots,\alpha_i)$$ and that $\{F^{(l)}(\alpha_1,\cdots,\alpha_i)\times W^{(l)}(\alpha_1,\cdots,\alpha_i)\,|\,\alpha_1,\cdots,\alpha_i;i\}$ is a covering of $X\times Y$.

From the relation that $F^{(l)}(\alpha_1,\cdots,\alpha_i)\subset G^{(l)}(\alpha_1,\cdots,\alpha_i;1)\cup G^{(l)}(\alpha_1,\cdots,\alpha_i;2)$ it follows that there are closed sets $F^{(l)}(\alpha_1,\cdots,\alpha_i;1)$ and $F^{(l)}(\alpha_1,\cdots,\alpha_i;2)$ of X such that $F^{(l)}(\alpha_1,\cdots,\alpha_i)=F^{(l)}(\alpha_1,\cdots,\alpha_i;1)\cup F^{(l)}(\alpha_1,\cdots,\alpha_i;2)$ and $F^{(l)}(\alpha_1,\cdots,\alpha_i;k)\subset G^{(l)}(\alpha_1,\cdots,\alpha_i;k)$ $(k=1,2)$.

The relation $(G^{(l)}(\alpha_1,\cdots,\alpha_i;1)\times W^{(l)}(\alpha_1,\cdots,\alpha_i))\cap F^{(l)}=\phi$ is reduced to $(F^{(l)}(\alpha_1,\cdots,\alpha_i;1)\times W^{(l)}(\alpha_1,\cdots,\alpha_i))\cap F^{(l)}=\phi$.

By the assumption that $\dim X=m$, there is a family

$$\{H^{(l)}(\alpha_1,\cdots,\alpha_i;k)\,|\,\alpha_1,\cdots,\alpha_i;i;1\leqq l\leqq s\}$$

of open sets in X such that $F^{(l)}(\alpha_1,\cdots,\alpha_i;k)\subset H^{(l)}(\alpha_1,\cdots,\alpha_i;k)\subset G^{(l)}(\alpha_1,\cdots,\alpha_i;k)$ and that

(2) the order of $\{\mathfrak{B}H^{(l)}(\alpha_1,\cdots,\alpha_i;k)\,|\,\alpha_1,\cdots,\alpha_i;i;1\leqq l\leqq s;k=1,2\}$ is at most m.

Let us put

(3) $H_i^{(l)}=\bigcup\{H^{(l)}(\alpha_1,\cdots,\alpha_i;2)\times W^{(l)}(\alpha_1,\cdots,\alpha_i)\,|\,\alpha_1,\cdots,\alpha_i\}$,

(4) $K_i^{(l)}=\bigcup\{H^{(l)}(\alpha_1,\cdots,\alpha_i;1)\times W^{(l)}(\alpha_1,\cdots,\alpha_i)\,|\,\alpha_1,\cdots,\alpha_i\}$.

And put $P_1^{(l)}=H_1^{(l)}$, $Q_1^{(l)}=K_1^{(l)}-\overline{H_1^{(l)}}$, $P_i^{(l)}=H_i^{(l)}-\bigcup_{j=1}^{i-1}\overline{K_j^{(l)}}$, $Q_i^{(l)}=K_i^{(l)}-\bigcup_{j=1}^{i}\overline{H_j^{(l)}}$ $(i\geqq 2)$, $P^{(l)}=\bigcup_{i=1}^{\infty}P_i^{(l)}$ and $Q^{(l)}=\bigcup_{i=1}^{\infty}Q_i^{(l)}$.*)

Then we have

(5) $$X\times Y=(\bigcup_{i=1}^{\infty}\overline{P_i^{(l)}})\cup(\bigcup_{i=1}^{\infty}\overline{Q_i^{(l)}}),$$

(6) $P^{(l)}\cap Q^{(l)}=\phi$, $\overline{P_j^{(l)}}\subset G^{(l)}$ $(j=1,2,\cdots)$ and $Q^{(l)}\cap\overline{M^{(l)}}=\phi$.

Finally we put $V^{(l)}=X\times Y-\overline{Q^{(l)}}$; then we have

(7) $$\mathfrak{B}V^{(l)}\subset\mathfrak{B}Q^{(l)}.$$

Since $Q^{(l)}\cap M^{(l)}=\phi$ by (6) and $M^{(l)}$ is open, we have $\overline{Q^{(l)}}\cap M^{(l)}=\phi$, and hence $F^{(l)}\subset M^{(l)}\subset V^{(l)}$.

On the other hand, since $V^{(l)}=X\times Y-\overline{Q^{(l)}}\subset X\times Y-\bigcup_{i=1}^{\infty}\overline{Q_i^{(l)}}\subset\bigcup_{i=1}^{\infty}\overline{P_i^{(l)}}\subset G^{(l)}$ we have

(8) $$F^{(l)}\subset V^{(l)}\subset G^{(l)}.$$

Since $\overline{P_i^{(l)}}=P_i^{(l)}\cup(\overline{P_i^{(l)}}-P_i^{(l)})$ and $\overline{Q_i^{(l)}}=Q_i^{(l)}\cup(\overline{Q_i^{(l)}}-Q_i^{(l)})$ we have by (5)

(9) $$X\times Y=P^{(l)}\cup Q^{(l)}\cup(\bigcup_{i=1}^{\infty}\mathfrak{B}P_i^{(l)})\cup(\bigcup_{i=1}^{\infty}\mathfrak{B}Q_i^{(l)}).$$

Since $P^{(l)}$ is open, $P^{(l)}\cap\overline{Q^{(l)}}=\phi$ by (6), and hence

(10) $$P^{(l)}\cap\mathfrak{B}Q^{(l)}=\phi.$$

Combining (7) with (9) and (10), we have

(11) $$\mathfrak{B}V^{(l)}\subset\mathfrak{B}Q^{(l)}\subset(\bigcup_{i=1}^{\infty}\mathfrak{B}P_i^{(l)})\cup(\bigcup_{i=1}^{\infty}\mathfrak{B}Q_i^{(l)}).$$

On the other hand we have

(12) $$\mathfrak{B}P_i^{(l)}=\mathfrak{B}(H_i^{(l)}-\bigcup_{j=1}^{i-1}\overline{K_j^{(l)}})\subset(\mathfrak{B}H_i^{(l)}\cup(\bigcup_{j=1}^{i-1}\mathfrak{B}K_j^{(l)})),$$

and

(13) $$\mathfrak{B}Q_i^{(l)}\subset(\mathfrak{B}K_i^{(l)}\cup(\bigcup_{j=1}^{i}\mathfrak{B}H_j^{(l)})).$$

Now (11), (12), and (13) give us

(14) $$\mathfrak{B}V^{(l)}\subset\mathfrak{B}Q^{(l)}\subset(\bigcup_{i=1}^{\infty}\mathfrak{B}H_i^{(l)})\cup(\bigcup_{i=1}^{\infty}\mathfrak{B}K_i^{(l)}),$$

hence we have

(15) $$\bigcap_{l=1}^{s}\mathfrak{B}V^{(l)}\subset\bigcap_{l=1}^{s}[(\bigcup_{i=1}^{\infty}\mathfrak{B}H_i^{(l)})\cup(\bigcup_{i=1}^{\infty}\mathfrak{B}K_i^{(l)})].$$

Since $\{W^{(l)}(\alpha_1,\cdots,\alpha_i)\,|\,\alpha_1,\cdots,\alpha_i\}$ is locally finite, we have

*) The argument below is the same as that in [6, Lemma 2.2].

$$\mathfrak{B}H_i^{(l)}\subset[\bigcup\{\mathfrak{B}(H^{(l)}(\alpha_1,\cdots,\alpha_i;2)\times W^{(l)}(\alpha_1,\cdots,\alpha_i))\mid\alpha_1,\cdots,\alpha_i\}]$$
$$=[\bigcup\{\mathfrak{B}H^{(l)}(\alpha_1,\cdots,\alpha_i;2)\times\overline{W^{(l)}(\alpha_1,\cdots,\alpha_i)}\mid\alpha_1,\cdots,\alpha_i\}]$$
$$\cup[\bigcup\{\overline{H^{(l)}(\alpha_1,\cdots,\alpha_i;2)}\times\mathfrak{B}W^{(l)}(\alpha_1,\cdots,\alpha_i)\mid\alpha_1,\cdots,\alpha_i\}]$$

in view of (3). Likewise

$$\mathfrak{B}K_i^{(l)}\subset[\bigcup\{\mathfrak{B}H^{(l)}(\alpha_1,\cdots,\alpha_i;1)\times\overline{W^{(l)}(\alpha_1,\cdots,\alpha_i)}\mid\alpha_1,\cdots,\alpha_i\}]$$
$$\cup[\bigcup\{H^{(l)}(\alpha_1,\cdots,\alpha_i;1)\times\mathfrak{B}W^{(l)}(\alpha_1,\cdots,\alpha_i)\mid\alpha_1,\cdots,\alpha_i\}]$$

in view of (4).

Let us put $E_1^{(l)}(\alpha_1,\cdots,\alpha_i;k)=\mathfrak{B}H^{(l)}(\alpha_1,\cdots,\alpha_i;k)\times\overline{W^{(l)}(\alpha_1,\cdots,\alpha_i)}$ and $E_2^{(l)}(\alpha_1,\cdots,\alpha_i;k)=\overline{H^{(l)}(\alpha_1,\cdots,\alpha_i;k)}\times\mathfrak{B}W^{(l)}(\alpha_1,\cdots,\alpha_i)$.

Then (14) can be expressed as

(14′) $\mathfrak{B}V^{(l)}\subset\bigcup\{E_1^{(l)}(\alpha_1,\cdots,\alpha_i;k)\cup E_2^{(l)}(\alpha_1,\cdots,\alpha_i;k)\mid\alpha_1,\cdots,\alpha_i;i;$
$k=1,2\}$.

The right hand side of (15) is a union of sets of the form

(16) $$E_{\delta_1}^{(1)}(\alpha_1^{(1)},\cdots,\alpha_{i_1}^{(1)};k_1)\cap E_{\delta_2}^{(2)}(\alpha_1^{(2)},\cdots,\alpha_{i_2}^{(2)};k_2)\cap\cdots\cap E_{\delta_s}^{(s)}(\alpha_1^{(s)},\cdots,\alpha_{i_s}^{(2)};k_s)$$

in view of (14′). Here each of $\delta_1,\cdots,\delta_s,k_1,\cdots,k_s$ is 1 or 2.

Let us suppose that

$$\delta_{i_1}=\delta_{i_2}=\cdots=\delta_{i_p}=1,\ \delta_{j_1}=\delta_{j_2}=\cdots=\delta_{j_q}=2\quad(p+q=s).$$

If $p\geqq n+1$ then (16) is empty by (2), on the contrary if $p<n+1$ then $q=s-p=(m+n+1)-p\geqq m+1$ and (16) is also empty by Lemma 1. Thus (16) is empty in any case.

Consequently the right hand side of (15) is empty, hence we have

(17) $$\bigcap_{l=1}^{s}\mathfrak{B}V^{(l)}=\phi.$$

Now the theorem follows from (8) and (17) ([4]).

2. If X is perfectly normal, then $X\times Y$ is perfectly normal ([5]), and hence countably paracompact normal. The following Corollary 1 follows directly from the theorem.

Corollary 1. *If X is a perfectly normal space and Y is separable metrizable, then*

$$\dim(X\times Y)\leqq\dim X+\dim Y.$$

If Y is a countable union of locally compact subsets then $X\times Y$ is countably paracompact normal for any countably paracompact normal space X ([3]), hence the following Corollary 2 follows.

Corollary 2. *If X is a countably paracompact normal space and Y is a separable metrizable space which is a countable union of locally compact subsets, then*

$$\dim(X\times Y)\leqq\dim X+\dim Y.$$

3. Finally we shall show that Michael's space defined in [1] serves as a counter example for (A).

Let X be a topological space which is obtained from the closed unit interval [0, 1] by retopologizing it so that a set M is open if and only if M is expressed as $M=G\cup L$ with an open set G in the usual sense and with L consisting of irrationals. Then $\dim X=0$.

To show this we may construct such an open refinement of a given covering $\{U_1, U_2,\cdots, U_k\}$ that its order is 1.

Let $\{p_1, p_2,\cdots\}$ be the set of all rationals; then there is an interval (λ_i, μ_i) (λ_i and μ_i are irrational numbers in $[0,1]$) for any i which is a neighborhood of p_i and which is contained in one of $\{U_j\}$. Then $\{(\lambda_i, \mu_i)\,|\,i=1,2,\cdots\}$ is a family of open subsets of X each of which is contained in some element of $\{U_j\}$.

Now

$$(\lambda_i, \mu_i)-\bigcup_{t<i}[\lambda_t, \mu_t]$$

is expressed as a disjoint union of finite open intervals whose end points are irrational numbers. Accordingly

$$\{p_j\,|\,j=1, 2,\cdots\}\subset\sum_{i=1}^{\infty}(\alpha_i, \beta_i),$$

where α_i and β_i are irrational numbers and $\sum$ stands for a disjoint union.

Let us put

$$A=X-\sum_{i=1}^{\infty}(\alpha_i, \beta_i),$$

then A is a subset of irrationals.

We put

$$V_j=\bigcup\{(\alpha_i, \beta_i)\,|\,(\alpha_i, \beta_i)\not\subset U_l \text{ for } l<j, (\alpha_i, \beta_i)\subset U_j\}$$
$$\cup\{x\,|\,x\in A, x\notin U_l \text{ for } l<j, x\in U_j\}.$$

Clearly $\{V_1, V_2,\cdots, V_k\}$ is a disjoint family and it is the desired refinement of $\{U_1, U_2,\cdots, U_k\}$, and $\dim X=0$ follows.

Let Y be a subspace of closed interval $[0,1]$ consisting of all irrationals. Then, as is well known, $\dim Y=0$.

E. Michael has shown ([1]) that $X\times Y$ is not normal. Generally any 0-dimensional space is always normal, hence $X\times Y$ is not 0-dimensional, and hence

$$\dim X\times Y>\dim X+\dim Y.$$

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