59. A Note on the Convergence of Semi-groups of Operators

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(Comm. by Kinjirô KUNUGI, M.J.A., April 13, 1964)

1. In the following we shall deal with a sequence of one-parameter semi-groups $\{U_t^{(n)}\}$ $(t\geqq 0, n=1,2,\cdots)$ of operators on a fixed Banach space $\mathfrak{B}$ to $\mathfrak{B}$ which satisfies the stability condition, that is,

$$U_t^{(n)}U_{t'}^{(n)}=U_{t+t'}^{(n)}\quad (t,t'\geqq 0),\ U_0^{(n)}=I,$$
$$\lim_{t\to t_0} U_t^{(n)}f=U_{t_0}^{(n)}f\quad (t_0\geqq 0, f\in\mathfrak{B}),$$
$$||U_t^{(n)}||\leqq Me^{\alpha t},$$

where M and α are independent of n and t.

For simplicity we assume $M=1$.

Let $\mathfrak{G}^{(n)}$ be the infinitesimal generator of $\{U_t^{(n)}\}$, that is,

$$\mathfrak{G}^{(n)}\varphi=\lim_{h\downarrow 0} h^{-1}(U_h^{(n)}-I)\varphi,$$

then the domain $\mathfrak{D}(\mathfrak{G}^{(n)})$ of $\mathfrak{G}^{(n)}$ is dense in $\mathfrak{B}$, and for any $m>\alpha$ the inverse operator $I_m^{(n)}=(I-m^{-1}\mathfrak{G}^{(n)})^{-1}$ is linear and satisfies following relations

$$I_m^{(n)}f=m\int_0^\infty e^{-mt}U_t^{(n)}f\,dt\quad (f\in\mathfrak{B}),$$
$$||I_m^{(n)}||\leqq(1-m^{-1}\alpha)^{-1}.$$

Our aim is to solve the problem of the following type.

Assumption (A). *$\{\mathfrak{G}^{(n)}\varphi\}_n$ is a Cauchy sequence in $\mathfrak{B}$ for any $\varphi\in\mathfrak{M}\subseteqq\bigcup_k\bigcap_{n\geqq k}\mathfrak{D}(\mathfrak{G}^{(n)})$, where $\mathfrak{M}$ is dense in $\mathfrak{B}$.*

Under Assumption (A), is it true that the additive operator $\mathfrak{G}=\lim_{n\to\infty}\mathfrak{G}^{(n)}$ or some closed extension of $\mathfrak{G}$ is the infinitesimal generator of a semi-group $\{U_t\}$ which satisfies $U_t=\lim_{n\to\infty}U_t^{(n)}$?

Our main theorem Theorem 2 is an answer to this problem.

The following theorem had been treated by H. F. Trotter [1].

Theorem 1. *Under Assumption* (A), *the closure $\widetilde{\mathfrak{G}}$ of $\mathfrak{G}$ is the infinitesimal generator of a semi-group $\{U_t\}$ which satisfies $U_t=\lim_{n\to\infty}U_t^{(n)}$ if and only if the following Condition* (A_1) *is satisfied.*

Condition (A_1). *For some $m>\alpha$, the range $\mathfrak{R}(I-m^{-1}\mathfrak{G})$ of $I-m^{-1}\mathfrak{G}$ is dense in $\mathfrak{B}$.*

As an application we shall treat this theorem from above general point of view and prove Theorem 1 by using Theorem 2.

The author wishes to express his gratitude to Prof. S. Tsurumi

and Mr. K. Sato for their kind advice.

2. In the following we assume Assumption (A) and shall prove that Condition (B_1) gives a necessary and sufficient condition for this problem.

Condition (B_1). *For any* $t, t' \geqq 0$,

$$\lim_{n,n'\to\infty} \| U_t^{(n)} U_{t'}^{(n')} f - U_{t'}^{(n')} U_t^{(n)} f \| = 0 \quad (f \in \mathfrak{B}).$$

In this section we assume Condition (B_1).

Remark 1. *For any* $m, m' > \alpha$,

$$\lim_{n,n'\to\infty} \| I_m^{(n)} I_{m'}^{(n')} f - I_{m'}^{(n')} I_m^{(n)} f \| = 0 \quad (f \in \mathfrak{B}).$$

This assertion readily follows from (B_1).

Next, we shall prove the basic lemma.

Lemma. *For* any $m > \alpha$ *and* $f \in \mathfrak{B}$, $\{I_m^{(n)} f\}_n$ *is a Cauchy sequence in* $\mathfrak{B}$.

Proof. For any fixed $m > \alpha$ and $\varphi \in \mathfrak{M}$, we have

$$\begin{aligned}
&\| I_m^{(n)}(I - m^{-1}\mathfrak{G})\varphi - \varphi \| \\
&= \| I_m^{(n)}(I - m^{-1}\mathfrak{G})\varphi - I_m^{(n)}(I - m^{-1}\mathfrak{G}^{(n)})\varphi \| \\
&\leqq m^{-1} \| I_m^{(n)} \| \, \| \mathfrak{G}^{(n)}\varphi - \mathfrak{G}\varphi \| \\
&\leqq (m-\alpha)^{-1} \| \mathfrak{G}^{(n)}\varphi - \mathfrak{G}\varphi \|.
\end{aligned}$$

$$\therefore \lim_{n,n'\to\infty} \| I_m^{(n)}(I - m^{-1}\mathfrak{G})\varphi - I_m^{(n')}(I - m^{-1}\mathfrak{G})\varphi \| = 0.$$

Next, we shall prove that, for any $m_1 > 2^{-1}(m+\alpha)$,

$$\lim_{n,n'\to\infty} \| I_m^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| = 0.$$

By using the resolvent equation, we have

$$\begin{aligned}
&\| I_m^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\leqq m_1^{-1} | m_1 - m | \, \| I_m^{(n)} I_{m_1}^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\quad + m_1^{-1} m \| I_{m_1}^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \|.
\end{aligned}$$

Here

$$\begin{aligned}
&\| I_m^{(n)} I_{m_1}^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\leqq \| I_m^{(n)} I_{m_1}^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n)} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\quad + \| I_m^{(n)} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\leqq (1 - m^{-1}\alpha)^{-1} \| I_{m_1}^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\quad + \| I_m^{(n)} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \|,
\end{aligned}$$

and

$$\begin{aligned}
&\| I_m^{(n)} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\leqq \| I_m^{(n)} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi - I_{m_1}^{(n')} I_m^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\quad + \| I_{m_1}^{(n')} I_m^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_{m_1}^{(n')} I_m^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\leqq \| I_m^{(n)} I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi - I_{m_1}^{(n')} I_m^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\quad + (1 - m_1^{-1}\alpha)^{-1} \| I_m^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \|.
\end{aligned}$$

Hence, we obtain the desired inequality

$$\begin{aligned}
&\| I_m^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_m^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \| \\
&\leqq \frac{m_1 - \alpha}{m_1 - \alpha - |m_1 - m|} \cdot \frac{m - \alpha + |m_1 - m|}{m_1(1 - m^{-1}\alpha)} \| I_{m_1}^{(n)}(I - m_1^{-1}\mathfrak{G})\varphi - I_{m_1}^{(n')}(I - m_1^{-1}\mathfrak{G})\varphi \|
\end{aligned}$$

$$+\frac{m_1-\alpha}{m_1-\alpha-|m_1-m|}\cdot\frac{|m_1-m|}{m_1}||I_{m_1}^{(n')}I_m^{(n)}(I-m_1^{-1}\mathfrak{G})\varphi-I_m^{(n)}I_{m_1}^{(n')}(I-m_1^{-1}\mathfrak{G})\varphi||.$$

By virtue of Remark 1, for any $m_1>2^{-1}(m+\alpha)$, there exists $g\in\mathfrak{B}$ such that $\lim_{n\to\infty}I_m^{(n)}(I-m_1^{-1}\mathfrak{G})\varphi=g$. We define

$$\mathfrak{R}_m=\bigcup_{m_1>2^{-1}(m+\alpha)}\bigcup_{\varphi\in\mathfrak{M}}\{f;f=(I-m_1^{-1}\mathfrak{G})\varphi\}$$

Then $\overline{\mathfrak{R}}_m=\mathfrak{B}$.
Thus we have proved that for any $m>\alpha$ and $f\in\mathfrak{B}$, there exists $g_m\in\mathfrak{B}$ such that $\lim_{n\to\infty}I_m^{(n)}f=g_m$. We define $I_mf=g_m$ for $f\in\mathfrak{B}$.

Theorem 2. *Under Condition* (B_1), *we can construct a closed extension* $\widetilde{\mathfrak{G}}$ *of* $\mathfrak{G}$ *whose domain* $\mathfrak{D}(\widetilde{\mathfrak{G}})=\mathfrak{R}\supset\mathfrak{M}$,

$$\widetilde{\mathfrak{G}}\varphi=\lim_{h\downarrow 0}h^{-1}(U_h-I)\varphi\quad(\varphi\in\mathfrak{R}),$$

where $\{U_t\}$ *is a semi-group obtained by*

$$U_tf=\lim_{m\to\infty}\exp(t\widetilde{\mathfrak{G}}\,I_m)f\quad(f\in\mathfrak{B}).$$

Moreover $\{U_t\}$ *satisfies* $U_t=\lim_{n\to\infty}U_t^{(n)}$.

Proof. In the proof of Lemma, we have obtained $\mathfrak{D}(I_m)=\mathfrak{B}$ and $I_m(I-m^{-1}\mathfrak{G})\varphi=\varphi$.
It readily follows that $\mathfrak{R}(I_m)\supset\mathfrak{M}$, the additivity and the boundedness of $I_m(||I_m||\leqq(1-m^{-1}\alpha)^{-1})$.

Letting $n\to\infty$ in the resolvent equation for $I_m^{(n)}$, we have

$$I_mf=m_1^{-1}(m_1-m)I_mI_{m_1}f+m_1^{-1}mI_{m_1}f,$$
$$I_mI_{m_1}f=I_{m_1}I_mf.$$

Moreover we have

$$\lim_{m\to\infty}I_mf=f\quad(f\in\mathfrak{B}),$$

since

$$||I_m\varphi-\varphi||\leqq m^{-1}||I_m\mathfrak{G}\varphi||\leqq(m-\alpha)^{-1}||\mathfrak{G}\varphi||\quad(\varphi\in\mathfrak{M}),$$

and $\overline{\mathfrak{M}}=\mathfrak{B}$.

Now we show that I_m is a one-to-one transformation on $\mathfrak{B}$ to its range $\mathfrak{R}(I_m)$. For any f, $f'\in\mathfrak{B}$ such that $I_mf=I_mf'$, we have, using the resolvent equation

$$I_mf=m_1^{-1}(m_1-m)I_{m_1}I_mf+m_1^{-1}mI_{m_1}f,$$
$$I_mf'=m_1^{-1}(m_1-m)I_{m_1}I_mf'+m_1^{-1}mI_{m_1}f'.$$
$$\therefore\quad I_{m_1}f=I_{m_1}f'\quad(m_1>\alpha),$$

and letting $m_1\to\infty$, we have $f=f'$.

Since the resolvent equation shows that $\mathfrak{R}(I_m)=\mathfrak{R}(I_{m'})=\mathfrak{R}$ for any m,m', we have the inverse operator I_m^{-1} on $\mathfrak{R}$ to $\mathfrak{B}$.

We define the additive operator $\widetilde{\mathfrak{G}}_m$,

$$\widetilde{\mathfrak{G}}_m=m(I-I_m^{-1}).$$

We shall prove that $\widetilde{\mathfrak{G}}_m$ is independent of m. For any $f\in\mathfrak{B}$.
$\widetilde{\mathfrak{G}}_mI_mf=m(I-I_m^{-1})I_mf=m(I_m-I)f$.

On the other hand

$$\widetilde{\mathfrak{G}}_{m_1} I_m f = m_1(I - I_{m_1}^{-1}) I_m f = m_1 I_m f - m_1[m_1^{-1}(m_1 - m) I_m f + m_1^{-1} m f].$$

$$\therefore \quad \widetilde{\mathfrak{G}}_m = \widetilde{\mathfrak{G}}_{m_1} = \widetilde{\mathfrak{G}}.$$

Since additive operator $\widetilde{\mathfrak{G}}$ whose domain is dense in $\mathfrak{B}$ has linear operators $\{I_m = (I - m^{-1}\widetilde{\mathfrak{G}})^{-1}\}$ on $\mathfrak{B}$ to $\mathfrak{R}$ which satisfy $||I_m|| \leqq (1 - m^{-1}\alpha)^{-1}$, by the characterization theorem for the infinitesimal generator, there exists a semi-group $\{U_t\}$

$$U_t f = \lim_{m\to\infty} \exp(t\widetilde{\mathfrak{G}} I_m) f \quad (f \in \mathfrak{B}),$$

such that

$$\lim_{h\downarrow 0} h^{-1}(U_h - I)\varphi = \widetilde{\mathfrak{G}}\varphi \quad (\varphi \in \mathfrak{R}).$$

Since we have proved that for any $\varphi \in \mathfrak{M}$

$$I_m(I - m^{-1}\mathfrak{G})\varphi = \varphi,$$

if readily follows that $\widetilde{\mathfrak{G}}$ is a closed extension of $\mathfrak{G}$.

The proof of the last part of this theorem is the same as that of [1; Theorem 5.1].

3. **Remark 2.** *Under Assumption* (A), *the following conditions are mutually equivalent.*

$$(\mathrm{B}_1) \quad \lim_{n,n'\to\infty} ||U_t^{(n)} U_{t'}^{(n')} f - U_{t'}^{(n')} U_t^{(n)} f|| = 0 \quad (t, t' \geqq 0,\ f \in \mathfrak{B})$$

$$(\mathrm{B}_2) \quad \lim_{n,n'\to\infty} ||I_m^{(n)} I_{m'}^{(n')} f - I_{m'}^{(n')} I_m^{(n)} f|| = 0 \quad (m, m' > \alpha,\ f \in \mathfrak{B})$$

$$(\mathrm{B}_3) \quad \lim_{n,n'\to\infty} ||U_t^{(n)} f - U_t^{(n')} f|| = 0 \quad (t \geqq 0,\ f \in \mathfrak{B})$$

$$(\mathrm{B}_4) \quad \lim_{n,n'\to\infty} ||I_m^{(n)} f - I_m^{(n')} f|| = 0 \quad (m > \alpha,\ f \in \mathfrak{B})$$

Proof. By Remark 1, Lemma and Theorem 2, $(\mathrm{B}_1)\Rightarrow(\mathrm{B}_2)\Rightarrow(\mathrm{B}_4)\Rightarrow(\mathrm{B}_3)$ is obvious.

$(\mathrm{B}_3)\Rightarrow(\mathrm{B}_1)$: we define $U_t f = \lim_{n\to\infty} U_t^{(n)} f$, then

$$\begin{aligned}
&||U_t^{(n)} U_{t'}^{(n')} f - U_{t'}^{(n')} U_t^{(n)} f|| \\
&\leqq ||U_t^{(n)} U_{t'}^{(n')} f - U_t^{(n)} U_{t'} f|| + ||U_t^{(n)} U_{t'} f - U_t U_{t'} f|| \\
&+ ||U_t U_{t'} f - U_{t'} U_t f|| + ||U_{t'} U_t f - U_{t'}^{(n')} U_t f|| + ||U_{t'}^{(n')} U_t f + U_{t'}^{(n')} U_t^{(n)} f||.
\end{aligned}$$

Here

$$\begin{aligned}
&||U_t U_{t'} f - U_{t'} U_t f|| \\
&\leqq ||U_t U_{t'} f - U_t^{(n)} U_{t'} f|| + ||U_t^{(n)} U_{t'} f - U_t^{(n)} U_{t'}^{(n)} f|| \\
&+ ||U_{t'}^{(n)} U_t^{(n)} f - U_{t'}^{(n)} U_t f|| + ||U_{t'}^{(n)} U_t f - U_{t'} U_t f||.
\end{aligned}$$

$$\therefore \quad \lim_{n,n'\to\infty} ||U_t^{(n)} U_{t'}^{(n')} f - U_{t'}^{(n')} U_t^{(n)} f|| = 0 \quad (t, t' \geqq 0,\ f \in \mathfrak{B}).$$

It readily follows from Theorem 2 and Remark 2 that

Remark 3. *Under Assumption* (A), *there exists a closed extension* $\widetilde{\mathfrak{G}}$ *of* $\mathfrak{G}$ *which is the generator of a semi-group* $\{U_t\}$, *where* $U_t = \lim_{n\to\infty} U_t^{(n)}$, *if and only if Condition* (B_i) *is satisfied.*

Remark 4. *In Theorem 2, if the following Condition* (C) *is satisfied, then* $\widetilde{\mathfrak{G}}$ *is the closure of* $\mathfrak{G}$:

Condition (C). *There exists a subset* $\mathfrak{M}'\subset\mathfrak{B}$ *which is dense in* $\mathfrak{B}$ *such that* $I_m\mathfrak{M}'\subset\mathfrak{M}$ *for some* $m>\alpha$.

Proof. For any $f\in\mathfrak{B}$, we can choose a sequence $\{f_k\}\subset\mathfrak{M}'$ which converges to f as k tends to infinity. By the boundedness of I_m,

$$\lim_{k\to\infty}\|I_m f - I_m f_k\|=0,$$

where $I_m f_k\in\mathfrak{M}$ and

$$\|\tilde{\mathfrak{G}}\, I_m f - \mathfrak{G}\, I_m f_k\| \leqq [m+(m-\alpha)^{-1}m^2]\,\|f-f_k\|.$$

$$\therefore\ \lim_{k\to\infty}\|\tilde{\mathfrak{G}}\, I_m f - \mathfrak{G}\, I_m f_k\|=0,$$

which implies that $\tilde{\mathfrak{G}}$ is the closure of $\mathfrak{G}$.

Remark 5. *Proof of Theorem* 1.

First, we shall prove that (A_1) implies (B_4). It readily follows from the first part of the proof of Lemma that $\{I_m^{(n)}f\}_n$ is a Cauchy sequence in $\mathfrak{B}$ for some $m>\alpha$. Then the above assertion for any $m>\alpha$ can be proved by the same way as that of [1; Lemma 5.1].

$(A_1)\Rightarrow(C)$ can be proved by taking

$$\mathfrak{M}'=\bigcup_{\varphi\in\mathfrak{M}}\{f;\, f=(I-m^{-1}\mathfrak{G})\varphi\}.$$

Then Theorem 2, Remark 2 and Remark 4 show that the closure $\tilde{\mathfrak{G}}$ of $\mathfrak{G}$ is the infinitesimal generator of a semi-group $\{U_t\}$ which satisfiies $U_t=\lim_{n\to\infty}U_t^{(n)}$.

The inverse part of the theorem is obvious.

Remark 6. *Under Assumption* (A), *Condition* (B_i)+*Condition* (C) *is equivalent to Condition* (A_1).

References

[1] H. F. Trotter: Approximation of semi-groups of operators. Pacific Journal of Mathematics, **8**, 887–919 (1958).

[2] ——: On the product of semi-groups of operators. Proceedings of the American Mathematical Society, **10**, 545–551 (1959).