58. *On the Uniqueness of the Cauchy Problem for Semi-elliptic Partial Differential Equations.* III

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1. **Introduction.** In this note we shall remark the superfluity of the condition IV of the uniqueness theorems obtained in the previous note [5]. As Theorem 1 is fundamental among Theorems in [5], we shall only indicate the modifications to be done in its proof. That theorem is related as the following:

Theorem 1 in [5]. $P(x, D)=P_0(x, D)+Q(x, D)$,

$$P_0(x, D)=\sum_{|\alpha:m|=1} a_\alpha(x)D^\alpha,\ Q(x, D)=\sum_{j=1}^{n}\sum_{|\alpha:m|\leq 1-\frac{1}{m_j}} a_\alpha(x)D^\alpha.^{*)}$$

I. (1) $m_1\geq m_j$. (2) *The coefficients of* $P_0(x, D)$ *are in* $C^{2|m|}(\Omega)$ *and those of* $Q(x, D)$ *are in* $C(\Omega)$ *and bounded on* $\overline{\Omega}$, *where* Ω *is a domain containing* $x=0$. (3) *For* $\alpha=(m_1, 0,\cdots, 0)$, $a_\alpha(0)\neq 0$.

II. $P_0(x, D)$ *is semi-elliptic at* $x=0$, *i.e.* $P_0(0, \xi)$ *does not vanish for any non-zero real vector* ξ.

III. *Let* $\zeta_1=\zeta_1(\tilde{\xi})$ *be a root of* $P_0(0, \zeta_1, \tilde{\xi})=0$, *then* $P_0^{(1)}(0, \zeta_1, \tilde{\xi})$ *does not vanish for any non-zero real vector* $\tilde{\xi}$.

IV. Let be $N^0=(-1, 0,\cdots, 0)$, $N=(N_1, N_2,\cdots, N_n)$ where N_j's are real, and $\xi+i\tau N=(\xi_1+i\tau N_1,\cdots, \xi_n+i\tau N_n)$ where τ is a real number. For $m_1\geq 2$ there are neighborhoods $U_0(0)$ of $x=0$, $V_0(N^0)$ of N^0, and a constant C_0 such that

$$\sum_{j=1}^{n}\sum_{|\alpha:m|=1-\frac{1}{m_j}}|(\xi+i\tau N)^\alpha|^2\leq C_0\Big[\sum_{j=1}^{n}|P_0^{(j)}(x, \xi+i\tau N)|^2+1\Big] \tag{1.1}$$

holds for any $x\in U_0(0)$, any $N\in V_0(N)$ and any $(\xi, \tau)\in \Xi^n\times R^1$, $\tau\geq 1$.

Suppose that I, II, III *and* IV *hold. Then there exist the constants* $C, \delta_0>0$, $M\geq 1$, *and for any real number* τ, δ *satisfying* $\delta<\delta_0$, $\tau\delta>M$,

$$\sum_{|\alpha:m|\leq 1}[(1+\tau\delta^2)\tau]^{m_0\left(1-\frac{1}{m_1}-|\alpha:m|\right)}\tau\int|D^\alpha u|^2\exp(2\tau\varphi_\delta(x))\,dx \tag{1.2}$$
$$\leq C\int|P(x, D)u|^2\exp(2\tau\varphi_\delta(x))\,dx$$

holds if $u\in C_0^\infty(U_\delta(0))$, *where* $\varphi_\delta(x)$ *is* $(x_1-\delta)^2+\delta\sum_{j=2}^{n}x_j^2$ *and* $U_\delta(0)$ *is a neighborhood depending on* δ.

2. **The superfluity of the condition IV.** We first used the

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*) $\alpha=(\alpha_1, \alpha_2,\cdots, \alpha_n)\alpha_j$; integer ≥ 0, $m=(m_1, m_2,\cdots, m_n)m_j$; integer >0, $|\alpha:m|=\sum_{j=1}^{n}\frac{\alpha_j}{m_j}$. For the other notations, see [5].

condition IV to estimate the third term in the right hand side of the inequality (5.8) in [5] (p. 788);

$$\sum_{|\alpha:m|=1}\int |D^{\alpha}u_g|^2 \exp(2\tau\varphi_\delta)\,dx \leq D_2\int\Big[\,|P_0(x, D)u_g|^2+(\tau\delta)^2|P_0^{(1)}(x, D)u_g|^2 + (\tau\delta)\sum_{j=1}^{n}\sum_{|\alpha:m|=1-\frac{1}{m_j}}|Du_g|^2\Big]\exp(2\tau\varphi_\delta)\,dx.$$

However this term is an estimation of

$$\int |P_0^{(1)}(x, D)u_g - P_0^{(1)}(x_g, D)u_g|^2 \exp(2\tau\varphi_\delta)\,dx$$

on the support of u_g. So the above third term can be replaced by

$$D_2(\tau\delta)\int \sum_{|\alpha:m|=1-\frac{1}{m_1}}|Du_g|^2 \exp(2\tau\varphi_\delta)\,dx.$$

Then by using (4.9) in [5] (p. 785), for any α; $|\alpha:m|=1-\dfrac{1}{m_1}$ there exists at least one β; $|\beta:m|=1$ such that

$$\int |D^{\alpha}u_g|^2 \exp(2\tau\varphi_\delta)\,dx = C\tau^{-1}\int |Du_g|^2 \exp(2\tau\varphi_\delta)\,dx$$

holds.

Thus we get for a constant D

$$(\tau\delta)\sum_{|\alpha:m|=1-\frac{1}{m_1}}\int |Du_g|^2 \exp(2\tau\varphi_\delta))\,dx \leq D\delta\sum_{|\alpha:m|=1}\int |Du_g|^2 \exp(2\tau\varphi_\delta)\,dx.$$

By transfering this term in (5.8) from the right to the left, and by choosing δ small properly, we get for a constant D_2'

$$\sum_{|\alpha:m|=1}\int |Du_g|^2 \exp(2\tau\varphi_\delta)\,dx \leq D_2'\int[\,|P_0(x, D)u_g|^2+(\tau\delta)^2|P_0^{(1)}(x, D)u_g|^2] \times\exp(2\tau\varphi_\delta)\,dx.$$

Thus in this case we can avoid to use the condition IV.

Next we used the condition IV to prove the inequality (5.14) in [5];

$$\sum_{j=1}^{n}A_{1-\frac{1}{m_j}} \leq D(\tau\delta)^{-1}\Big[A + A_1^{\frac{1}{2}}\Big(\sum_{j=1}^{n}A_{1-\frac{1}{m_j}}\Big)^{\frac{1}{2}}\Big]$$

where A_s and A denote $\sum_{|\alpha:m|=s}\int |Du|^2 \exp(2\varphi_\delta)\,dx$ and $\int |P_0(x, D)u|^2 \times\exp(2\tau\varphi_\delta)\,dx$ respectively for $u\in C_0^\infty(\Omega)$.

Taking notice that we need the above inequality, to get (1.2), only for $u\in C_0(U_\delta(0))$, we can use (5.15) in [5];

$$\tau(1+\delta^2\tau)A_{1-\frac{1}{m_j}} \leq CA_1 \quad \text{for each } j,\ u\in C_0^\infty(U_\delta(0)) \text{ and a constant } C.$$

By this, we can calculate the following:

$$A_{1-\frac{1}{m_j}} = (A_{1-\frac{1}{m_j}})^{\frac{1}{2}}(A_{1-\frac{1}{m_j}})^{\frac{1}{2}} \leq C[\tau(1+\delta^2\tau)]^{-\frac{1}{2}}A_1^{\frac{1}{2}}(A_{1-\frac{1}{m_j}})^{\frac{1}{2}}$$

$$\begin{aligned}\sum_{j=1}^{n}A_{1-\frac{1}{m_j}} &\leq C'[\tau(1+\delta^2\tau)]^{-\frac{1}{2}}A_1^{\frac{1}{2}}\Big[\sum_{j=1}^{n}(A_{1-\frac{1}{m_j}})^{\frac{1}{2}}\Big]\\ &= 2C'[\tau(1+\delta^2\tau)]^{-\frac{1}{2}}A_1^{\frac{1}{2}}\Big[\sum_{j=1}^{n}A_{1-\frac{1}{m_j}}\Big]^{\frac{1}{2}}\\ &= C''[\tau(1+\delta^2\tau)]^{-\frac{1}{2}}\Big[A + A_1^{\frac{1}{2}}\Big(\sum_{j=1}^{n}A_{1-\frac{1}{m_j}}\Big)^{\frac{1}{2}}\Big]\\ &= C''(\tau\delta)^{-1}\Big[A + A_1^{\frac{1}{2}}\Big(\sum_{j=1}^{n}A_{1-\frac{1}{m_j}}\Big)^{\frac{1}{2}}\Big].\end{aligned}$$

Thus we get (5.14) in [5] without the condition IV. The superfluity has proved.

3. Typical examples satisfying the conditions I, II and III.

(1) In the case of $m_1=m_2=\cdots=m_n$ $P_0(x, D)$ satisfying I, II and III is *the elliptic operator* same as that treated by L. Hörmander (see [1]).

(2) *The heat operator* $P_0(D)=D_1^2+D_2^2+\cdots+D_{n-1}^2+iD_n^2$ satisfies I, II and III for $m_1=m_2=\cdots=m_{n-1}=2, m_n=1$.

(3) $P_0(x, D)=(iD_1)^n+a(x)D_2^2$, n; odd number $\geqq 3$, $a(x)>0$, $a(x) \in C^{n+2}(\Omega)$, satisfies I, II and III for $m_1=n$, $m_2=2$. This result is due to M. Picone (see [3] and [4]). He proved for any integer >2.

(4) $P_0(D)=(iD_1)^{m_1}+a(iD_2)^{m_2}$, $m_1>m_2$, one is odd, the other is even, a; a constant $\neq 0$, satisfies I, II and III. This result is due to L. Nireberg (see [2]). He proved the uniqueness without "odd, even" restriction on m_1 and m_2.

References

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