57 *Representation of the State Vectors by Gelfand's Construction*

By Hideo YAMAGATA

(Comm. by Kinjirô KUNUGI, M.J.A., April 13, 1964)

§1. **Introduction.** In the previous paper [1], we have pointed out that Wightman's method using Gelfand's construction can treat only the free field.

Here, at the first step, the definition of the extended exponential function is given. (In another paper, we will give the definition of the extended exponential function which is more general and more faithful to the local field theory.) Using this definition, let's show the following facts:

(1) If the testing function's space in the expresssion $\exp(i\varphi(f))$ or $\exp(i\pi(f))$ is $(\mathfrak{S})$, the extended exponential function is same as the ordinary exponential function and we can construct only the eigenvectors of free Hamiltonian by the above construction.

Namely, using this cut-off of momenta $\boldsymbol{k}$, the extended exponential function can be reduced to the ordinary exponential function.

(2) If we wish to construct the eigenvectors of total Hamiltonian related to the field with interaction, at least the testing function δ as the element of the sequence space must be used. Namely, furthermore, the conditional convergence must be used [2].

§2. **Notations and definitions.** As the first step, the explicite form of the field functions $\varphi(\boldsymbol{x})$, $\pi(\boldsymbol{x})$, creation and annihilation operators $a^+(\boldsymbol{k})$, $a(\boldsymbol{k})$ and state vectors will be written down.

$$\varphi(\boldsymbol{x})=(1/(2\pi)^{3/2})\left\{\int a^+(\boldsymbol{k})\, e^{i\boldsymbol{k}\boldsymbol{x}}d\boldsymbol{k}+\int a(\boldsymbol{k})\, e^{-i\boldsymbol{k}\boldsymbol{x}}d\boldsymbol{k}\right\} \tag{1}$$

$$\pi(\boldsymbol{x})=(1/(2\pi)^{3/2})\left\{-\int ik_0 a^+(\boldsymbol{k})\, e^{i\boldsymbol{k}\boldsymbol{x}}d\boldsymbol{k}+\int ik_0 a(\boldsymbol{k})\, e^{-i\boldsymbol{k}\boldsymbol{x}}d\boldsymbol{k}\right\} \tag{2}$$

where $k_0=\sqrt{k_1^2+k_2^2+k_3^2+m^2}$.

Let's enclose the system in a box of finite volume V.

Considering the periodical extension of the above system, these formulas vary to the following formulas:

$$\varphi_V(\boldsymbol{x})=(1/\sqrt{V})[\textstyle\sum_{\boldsymbol{k}=(k_1,k_2,k_3)}(a^+(\boldsymbol{k})\, e^{i\boldsymbol{k}\boldsymbol{x}}+a(\boldsymbol{k})\, e^{-i\boldsymbol{k}\boldsymbol{x}})] \tag{3}$$

$$\pi_V(\boldsymbol{x})=(1/\sqrt{V})[\textstyle\sum_{\boldsymbol{k}=(k_1,k_2,k_3)}(-ik_0 a^+(\boldsymbol{k})\, e^{i\boldsymbol{k}\boldsymbol{x}}+ik_0 a(\boldsymbol{k})\, e^{-i\boldsymbol{k}\boldsymbol{x}})], \tag{4}$$

where $k_0=\sqrt{k_1^2+k_2^2+k_3^2+m^2}$, and k_1, k_2, k_3 are non-negative integers.

Hereafter, we use the following abbriviations: $a^+(\boldsymbol{k})\, e^{i\boldsymbol{k}\boldsymbol{x}}\equiv a^+(\boldsymbol{k}, \boldsymbol{x})$, $a(\boldsymbol{k})\, e^{-i\boldsymbol{k}\boldsymbol{x}}\equiv a(\boldsymbol{k}, \boldsymbol{x})$.

Creation and annihilation operators $a^+(\boldsymbol{k})$, $a(\boldsymbol{k})$ satisfy the conditions $[a^+(\boldsymbol{k}), a^+(\boldsymbol{k}')]=[a(\boldsymbol{k}), a(\boldsymbol{k}')]=0$ and $[a(\boldsymbol{k}), a^+(\boldsymbol{k}')]=\delta_{\boldsymbol{k},\boldsymbol{k}'}$.

Let's characterize the simultaneous eigenstate of the operators $N_k=a^+(\boldsymbol{k})a(\boldsymbol{k})$ for all $\boldsymbol{k}$ by non-negative integer valued function $n(\boldsymbol{k})$.

We denote this eigenstate with norm 1 by $\psi(n(\boldsymbol{k}))$. So the creation and annihilation operators can be represented by the formulas $a^+(\tilde{\boldsymbol{k}})\psi(n(\boldsymbol{k}))=\sqrt{n_{\tilde{\boldsymbol{k}}}+1}\psi(n'(\boldsymbol{k}))$ and $a(\tilde{\boldsymbol{k}})\psi(n(\boldsymbol{k}))=\sqrt{n_{\tilde{\boldsymbol{k}}}}\psi(n''(\boldsymbol{k}))$, where

$$n'(\boldsymbol{k})=\begin{cases} n(\boldsymbol{k}) & \text{for } \boldsymbol{k}\neq\tilde{\boldsymbol{k}} \\ n(\boldsymbol{k})+1 & \text{for } \boldsymbol{k}=\tilde{\boldsymbol{k}}\end{cases} \text{ and } n''(\boldsymbol{k})=\begin{cases} n(\boldsymbol{k}) & \text{for } \boldsymbol{k}\neq\tilde{\boldsymbol{k}} \\ n(\boldsymbol{k})-1 & \text{for } \boldsymbol{k}=\tilde{\boldsymbol{k}}.\end{cases}$$

$\psi(0(\boldsymbol{k}))$ corresponds to the vacuum state. Here we show the definition of the extended exponential function which play the most important role in this paper.

Let $a^h(\boldsymbol{k})$ denote the unbounded operator $a(\boldsymbol{k})$ for $h=0$ and $a^+(\boldsymbol{k})$ for $h=1$.

Let $b(a^h(\boldsymbol{k}))$ denote the coefficient of $a^h(\boldsymbol{k})$ in $\varphi(f)$.

Definition. *We denote by* $\overline{\exp(i\varphi(f))}\psi(0(\boldsymbol{k}))$ *the sum of the following state vectors*:

(1) $\prod_{j=1}^{n}(b(a^{h_j}(\boldsymbol{k}_j))a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j)\psi(0(\boldsymbol{k}))$ *where* $h_j=0$ *or* 1.

(2) (i) *Let* σ *denote the set of all possible operators* $\prod_{j=1}^{\infty}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\}$ *such that the state* $\psi(n(\boldsymbol{k}))$ *can be constructed from* $\psi(0(\boldsymbol{k}))$ *by using the creation and annihilation operators in this infinite product.*

(ii) *Let* σ_N *denote the set of all operators* $\prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\}$ *constructed from* $\prod_{j=1}^{\infty}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\}$ *in* σ.

(iii) *Let* $C(n(\boldsymbol{k}), \prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\})$ *denote the complex number* $\prod_{j=1}^{N}\{b(a^{h_j}(\boldsymbol{k}_j))/j\}\,||\prod_{j=1}^{N}a^{h_j}(\boldsymbol{k}_j)\psi(0(\boldsymbol{k}))||$. *If* $C(n(\boldsymbol{k}))=\lim_{N\to\infty}C(n(\boldsymbol{k}), \prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\})$ *is definite and non-zero, we say that the state* $C(n(\boldsymbol{k}))\psi(n(\boldsymbol{k}))$ *is also the component of the state* $\overline{\exp(i\varphi(f))}\psi(0(\boldsymbol{k}))$.

If $\varphi(\boldsymbol{x})$ can be expressed by the formulas (3) and (4), then $d\boldsymbol{k}_j=1$. Next, we show the correspondence between Von Neumann's direct product space and the space of state vectors $\psi(n(\boldsymbol{k}))$ [3].

We denote by $\mathfrak{K}$ the set of all possible momenta $\boldsymbol{k}$.

Von Neumann's direct product space is represented by $\prod_{\boldsymbol{k}\in\mathfrak{K}}\otimes\mathfrak{H}_{\boldsymbol{k}}$.

In this representation $\boldsymbol{k}$ corresponds to a momentum, the vacuum state $\psi(0(\boldsymbol{k}))$ corresponds to a $\prod_{\boldsymbol{k}\in\mathfrak{K}}\otimes\varphi_{0\boldsymbol{k}}$ ($\varphi_{0\boldsymbol{k}}\in\mathfrak{H}_{\boldsymbol{k}}$, $||\varphi_{0\boldsymbol{k}}||=1$).

Von Neumann's direct product space can be decomposed in the incomplete direct product space: $\prod_{\boldsymbol{k}\in\mathfrak{K}}\otimes\mathfrak{H}_{\boldsymbol{k}}=\mathfrak{H}^0\oplus\mathfrak{H}'\oplus\cdots$.

The bases of $\mathfrak{H}^0$ are written by $\prod_{\boldsymbol{k}\in\mathfrak{K}}\otimes\varphi_{n(\boldsymbol{k}),\boldsymbol{k}}$, where the non-negative integer $n(\boldsymbol{k})=0$ except for finite $\boldsymbol{k}$.

These $\mathfrak{H}^i$ have the following properties:

(i) $\mathfrak{H}^i\perp\mathfrak{H}^j$ for $1\neq j$,

(ii) $\mathfrak{H}^0$ contains the vacuum state $\psi(0(\boldsymbol{k}))$.

Specially we denote by $\mathfrak{H}_{\boldsymbol{k}}^0$ the subspace of Von Neumann's direct product space whose bases are $\{[a^+(\boldsymbol{k})]^n\psi(0(\boldsymbol{k}))/||[a^+(\boldsymbol{k})]^n\psi(0(\boldsymbol{k}))||;$

$n=0, 1, \cdots\}$.

§3. The effect of the cut-off.

Example 1. Let's choose the generalized function $(2\pi^{3/2}\mathfrak{F}^{-1}\delta=\{(2\pi^{3/2}\mathfrak{F}^{-1}\rho_{1/n}(x); n=1, 2, \cdots\}$ as f, where $\rho_{1/n}(x)$ is the function defined by L. Schwartz [5].

In this case, $\exp i\varphi(f)\psi(0(k))=\sum_{n=0}^{\infty}(1/n!)\,[i\{a(0)+a^+(0)\}]^n\psi(0(k))$. By the same way, $\exp it\varphi(f)\psi(0(k))=\sum_{n=0}^{\infty}(t^n/n!)[i\{a(0)+a^+(0)\}]^n\psi(0(k))$, where t is a parameter.

Since $\lim_{N\to\infty}\|\prod_{j=1}^{N}\{i(a(0)+a^+(0))t/j\}\psi(0(k))\|\leqq\lim_{N\to\infty}|\prod_{j=1}^{N}(2t/\sqrt{j})|=0$ for finite t, then $\overline{\exp it\varphi(f)}\psi(0(k))$ is the same as ordinary $\exp\varphi(f)\psi(0(k))$, and is sum of the infinite state vectors $(t^n/n!)a^{h_1}(k_1)a^{h_2}(k_2)\cdots a^{h_n}(k_n)\psi(0(k))$, where $a^{h_i}(k_i)$ is one of the operators $a(k_i)$ or $a^+(k_i)$ and n is a finite positive integer.

At the first step, $[\overline{\exp it\varphi(f)}\psi(0(k))]_{t=0}=\psi(0(k))$. Next the formal derivative of the state $\overline{\exp it\varphi(f)}\psi(0(k))$ by t is the following: $\lim_{t\to 0}[\{\exp it\varphi(f)\psi(0(k))-\psi(0(k))\}/it]=\lim_{t\to 0}[\sum_{n=1}^{\infty}(t^{n-1}/n!)i^{n-1}\{a(0)+a^+(0)\}^n\cdot\psi(0(k))]=\{a(0)+a^+(0)\}\cdot\psi(0(k))$ in $\prod_{k\in\mathfrak{K}}\otimes\mathfrak{H}_k$. By the same way, considering the finite difference of $\overline{\exp it\varphi(f)}\psi(0(k))$ corresponding to $[(1/i^n)\cdot d^n/dt^n\,\overline{\exp it\varphi(f)}]\psi(0(k))$, we obtain the sequence $[\psi_n]=[\sum_{i=1}^{n+1}C_i\,\overline{\exp it_{ni}\varphi(f)}\cdot\psi(0(k))]$, which converge to $\{a(0)+a^+(0)\}^n\cdot\psi(0(k))$ in $\prod_{k\in\mathfrak{K}}\otimes\mathfrak{H}_k$.

By the suitable linear sum of the above sequences, the sequence $[\psi_m]=[\sum_{i=1}^{N_m}C_i\,\overline{\exp it_{mi}\varphi(f)}\cdot\psi(0(k))]$ converging to an arbitrary element in $\mathfrak{H}_0^0$ is obtained.

Theorem. *If f is contained in* $(\mathfrak{S})$, *then* $\overline{\exp(i\varphi(f))}\psi(0(k))$ *is contained in* $\mathfrak{H}^0$.

Proof. $\exp(i\varphi(f))\psi(0(k))$ can be expressed by the formula

$$\exp(i\varphi(f))=\sum_{n=0}^{\infty}(i^n/n!)\Big[(1/(2\pi)^{3/2})\Big\{\iint\{f(x)a^+(k, x)+f(x)a(k, x)\}dkdx\Big\}\Big]^n\psi(0(k)).$$

Let $\psi^{(n)}(0(k))$ denote the state vector $(i^n/n!)\Big[(1/(2\pi)^{3/2})\times\Big\{\iint\{f(x)a^+(k, x)+f(x)a(k, x)\}dkdx\Big\}\Big]^n\psi(0(k))$. We can easily see that the inequality $\|\psi^{(n)}(0(k))\|^2\leqq(1/\{n!(2\pi)^{3n}\})\cdot\Big[2\int|f(x)|dx\Big]^{2n}$ is held. Since $\lim_{n\to\infty}\|\psi^{(n)}(0(k))\|^2\leqq\lim_{n\to\infty}(1/\{n!(2\pi)^{3n}\})\cdot\Big[2\int|f(x)|dx\Big]^{2n}=0$, $\overline{\exp(i\varphi(f))}\cdot\psi(0(k))$ is the same as $\exp(i\varphi(f))\psi(0(k))$, and is sum of the infinite orthogonal elements $\psi^{(n)}(0(k))$ $(n=0, 1, 2, \cdots)$. Furthermore, from the above inequality, it follows that for an arbitrary $\varepsilon>0$, there exists N such that the following inequality is held:

$\|\sum_{n\geq N}\psi^{(n)}(0(k))\|^2<\varepsilon$. Then the infinite linear sum $\overline{\exp i\varphi(f)}\cdot\psi(0(k))$ is convergent.

Secondary, let's treat the terms $\psi^{(n)}(0(k))$ for $n<N$. From the Fourier transform of $f(x)$ $F(k)=\mathfrak{F}f(x)$, let's construct the set of functions $\{F_{\tilde{N}}(k)\}$ depending to $\tilde{N}$ with the following properties: (1) $F_{\tilde{N}}(k)\}$ is contained in $(\mathfrak{S})$. (2) $F_{\tilde{N}}(k)=F(k)$ for $|k|\leqq\tilde{N}$. (3) $F_{\tilde{N}}(k)=0$ for $|k|>N+1$. (4) $\lim_{\tilde{N}\to\infty}F_{\tilde{N}}(k)=F(k)$ in the topology $(\mathfrak{S})$.

Let $f_{\tilde{N}}(x)$ denote the inverse Fourier transform $\mathfrak{F}^{-1}F_{\tilde{N}}(k)$.

Then $\lim_{\tilde{N}\to\infty}f_{\tilde{N}}(x)=f(x)$ in $(\mathfrak{S})$.

Since $f_{\tilde{N}}$ is contained in the space $\mathfrak{F}(D)=(Z)$ we can easily see that $\varphi(f_{\tilde{N}})=\int\varphi(x)f_{\tilde{N}}(x)dx=\Big[\int\varphi(x')f_{\tilde{N}}(x'-x)dx'\Big]_{x=0}=[\varphi*f_{\tilde{N}}]_{x=0}$. Since $\varphi*f_{\tilde{N}}=\mathfrak{F}^{-1}[(\mathfrak{F}\varphi\cdot\mathfrak{F}f_{\tilde{N}}(k)]$, the effect of the sufficiently large k does not appear in $\varphi(f_{\tilde{N}})$. Next $\|\{\varphi(f_{\tilde{N}})-\varphi(f)\}\varphi(0(k))\|^2=\Big\|\Big[(1/(2\pi)^{3/2})\times\Big\{\int\int(f(x)-f_{\tilde{N}}(x))(a^+(k,x)+a(k,x))dkdx\Big\}\Big]\cdot\psi(0(k))\Big\|^2\leqq(1/(2\pi)^3)\cdot\Big[2\int|f(x)-f_{\tilde{N}}(x)|\,dx\Big]^2$, and $\|[\{\varphi(f_{\tilde{N}})\}^n-\{\varphi(f)\}^n]\psi(0(k))\|^2=\|[\{\varphi(f_{\tilde{N}})\}^n-\varphi(f)\{\varphi(f_{\tilde{N}})\}^{n-1}+\cdots-\{\varphi(f)\}^n]\psi\|^2\leqq n\sqrt{n!}(1/(2\pi)^{3n})\Big[4\int|f(x)|\,dx\Big]^{2(n-1)}\cdot\Big[2\int|f(x)-f_{\tilde{N}}(x)|\,dx\Big]^2$ $(n=0,1,\cdots)$ for $f(x)$ with the property $\int|f(x)|\,dx>0$ and for sufficiently large $\tilde{N}$. (If $f(x)\equiv0$, the result of this theorem is obvious.) From the above inequalities, it follows that for an arbitrary $\varepsilon>0$, there exists $M>0$ such that $\|[\{\varphi(f_{\tilde{N}})\}^n-\{\varphi(f)\}^n]\psi(0(k))\|^2<\varepsilon$ for any pair $(n,\tilde{N})$ satisfied $0\leqq n<N$ and $\tilde{N}>M$.

Hence, for f contained in $(\mathfrak{S})$, $\overline{\exp(i\varphi(f))}\psi(0(k))$ is the limit of the sequence in $\mathfrak{H}^0$ and contained in $\mathfrak{H}^0$.

§4. **The conditional convergence.** In the following Example 2, it is obvious that the limit appearing in Definition is not convergent. But by varying the method of the decision of coefficient $C(n(k))$ in Definition and by using a sort of the conditional convergence, we can obtain a definite non-zero component of the state corresponding to that in Definition.

These components are not contained in $\mathfrak{H}^0$.

Furthermore, we can easily see that in §3 even if we use this method $\overline{\exp it\varphi(f)}\psi(0(k))$ is the same as ordinary $\exp it\varphi(f)\psi(0(k))$. Let's show here lemma using in the following Example 2.

Lemma. $|(\sqrt[2k]{k!})^p/\sqrt{p!}|<1$ *for* $p>k$,
$|(\sqrt[2k]{k!})^p/\sqrt{p!}|=1$ *for* $p=k$.

Proof. (2) is evident from this formula.

From $[({}^{2k}\sqrt{k!})^p/\sqrt{p!}]/[({}^{2k}\sqrt{k!})^{p-1}/\sqrt{(p-1)!}]={}^{2k}\sqrt{k!}/\sqrt{p}<1$ for $p>k$, the inequality (1) is obtained.

Example 2. Choose the generalized function $(2\pi)^{3/2}\,{}^{2k}\sqrt{k!}\,\delta_x=(-\pi/2, 0, 0)$ as the testing function f, then $\exp i\varphi(f)=\sum_{n=0}^{\infty}(i^n/n!)[{}^{2k}\sqrt{k!}\{\sum_{k=(k_1,k_2,k_3)}(i^{-k_1}a^+(\boldsymbol{k})+i^{k_1}a(\boldsymbol{k}))\}]^n$ for non-negative integers k_1, k_2, k_3.

Let's construct the component $\psi(1(4l+1, k_2, k_3), 0$ (for the other $\boldsymbol{k}$)) of the state vector $\overline{\exp i\varphi(f)}\psi(0(\boldsymbol{k}))$ not contained in $\mathfrak{H}^0$ by using a sort of conditional convergence.

Let $\tilde{\psi}$ denote this component. Ordering the set of the triplet of non negative integers $\{(4l+1, k_2, k_3); l, k_2, k_3$ are non negative integers$\}$, construct the following sequence (1, 0, 0), (1, 1, 0), (1, 0, 1), (1, 2, 0), (1, 1, 1), (1, 0, 2), (1, 3, 0), (1, 2, 1), (1, 0, 3), (1, 4, 0), (1, 3, 1), (1, 2, 2), (1, 1, 3), (1, 0, 4), (5, 0, 0), (5, 1, 0), $\cdots$.

Using the first m terms of the above sequence p-hold, construct the considerable many sequences. Then we have $(pm)!/(p!)^m$ sequences. Operate to the state $\psi(0(\boldsymbol{k}))$ pm creation operators $i^{-1}a^+(\boldsymbol{k})$ iteratedly as follows: $\{\prod_{i=1}^{pm}(i^{-1}a^+(\boldsymbol{k}_i))\}\psi(0(\boldsymbol{k}))$, where $\boldsymbol{k}_i$ correspond to the terms of one of the above $(pm)!/(p!)^m$ sequences.

Let $\sigma'_{pm,0}$ denote the set of the above constructed states corresponding to $(pm)!/(p!)^m$ sequences.

$$\sum_{\sigma'_{pm,0}}C(\{p(4l+1, k_2, k_3), 0 \text{ (for the other } \boldsymbol{k})\}, \prod_{j=1}^{pm}\{a^+(\boldsymbol{k}_j)/j\})=\{({}^{2k}\sqrt{k!})^p/\sqrt{p!}\}^m.$$

Since $|({}^{2k}\sqrt{k!})^p/\sqrt{p!}|<1$ for $p>k$ and $=1$ for $p=k$ according to the above Lemma, it follows that $\lim_{m\to\infty}\{({}^{2k}\sqrt{k!})^p/\sqrt{p!}\}^m=0$ for $p>k$ and $=1$ for $p=k$.

For $p=k=1$, this limit is a part of coefficient of the component $\tilde{\psi}$. Hereafter, let's confine the case $k=1$.

Next, by the same way, let's obtain the coefficient of the same state $\tilde{\psi}$ constructed from $\prod_{j=1}^{\infty}\{a^{h_j}(\boldsymbol{k}_j)/j\}\psi(0(\boldsymbol{k}))\neq 0$ (by means of condition convergence) in which only one $a^{h_j}(\boldsymbol{k}_j)$ is an annihilation operator. Consider the set $\sigma'_{N,1}$ of the all possible states $\prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)/j\}\psi(0(\boldsymbol{k}))\neq 0$ such that only one annihilation operator $a(\boldsymbol{k}_j)$ is always contained in this formula and the operators $a^{h_j}(\boldsymbol{k}_j)/j$ are always related to first $(N-2)$, $\boldsymbol{k}_j$ of the sequence which is obtained by the ordering of the set of the above triplet. Then the sum of the coefficients is as follows:

$$\begin{aligned}&\sum_{\sigma'_{N,1}}C(\{1(4l+1, k_2, k_3), 0 \text{(for the other } \boldsymbol{k})\}, \prod_{j-1}^{N}\{a^{h_j}(\boldsymbol{k}_j)/j\})\\&=(1/N!)\{2\cdot(N-2)!\cdot 1^N\cdot(-1)(N-2)\}(1+\sqrt{2})\\&=-2(N-2)/\{N(N-1)\}\times(1+\sqrt{2}).\end{aligned}$$

And $\lim_{N\to\infty}-2(N-2)/\{N(N-1)\}\times(1+2)=0$. Hence the desired coefficient is 0.

Finally let's obtain the coefficient of the same state $\widetilde{\psi}$ constructed by the form $\prod_{j=1}^{\infty}\{a^{h_j}(\boldsymbol{k}_j)/j\}\psi(0(\boldsymbol{k}))\neq 0$ in which only two $a^{h_j}(\boldsymbol{k}_j)$ are annihilation operators. Consider the set $\sigma'_{N,2}$ of the state $\prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)/j\}$ $\psi(0(\boldsymbol{k}))\neq 0$ such that only two annihilation operators are always contained in this formula and the operators $a^{h_j}(\boldsymbol{k}_j)/j$ are always related to first $(N-4)$, $\boldsymbol{k}_j$ of the above sequence.

Then $\sum_{\sigma'_{N,2}} C(\{1(4l+1, k_2, k_3),\ 0(\text{for the other } \boldsymbol{k})\},\ \prod_{j=1}^{N}\{a^{h_j}(/j\})$

$$=(N-4)\{15+2(N-5)\}/N(N-1)(N-2)(N-3).$$

$\lim_{N\to\infty}(N-4)\{15+2(N-5)\}/N(N-1)(N-2)(N-3)=0$. Hence the desired coefficient is 0. From the iteration of the same discussion, we obtain the component $\widetilde{\psi}\notin\mathfrak{H}^0$.

The above results depend on the rule of the decision of the coefficient. Perhaps this situation will be related to the sequences which is a part of δ. Because δ is considered as a condensation of infinitely many various properties [2]. At last, let's investigate the meaning of the component $\prod_{j=1}^{\infty}\{b(a^{h_j}(\boldsymbol{k}_j)a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\}\psi(0(\boldsymbol{k}))$.

Let's decompose $\varphi(f_n)$ in the following form: $\psi(f_n)=\int \lambda dE_{\varphi(f_n)}(\lambda)$, where $\lim_{n\to\infty} f_n=\delta$. So, $\exp i\varphi(f_n)=\int e^{i\lambda}dE_{\varphi(f_n)}(\lambda)$. From the following expample, it seems to us that this compornent is related to $\lambda=\pm\infty$ and the limit of the sequence $\{E_{\varphi(f_n)}(\lambda)\}$. About this problem we will give more precise discussion in another paper.

Example 3. Choose the generalized function $(2\pi)^{3/2}\delta_{\boldsymbol{x}=(0,0,0)}$ as the function f.

Then $\exp i\varphi(f)=\sum_{n=0}^{\infty}(i^n/n!)\,[\sum_{\boldsymbol{k}=(k_1,k_2,k_3)}(a^+(\boldsymbol{k})+a(\boldsymbol{k}))]^n$ for non negative integer k_1, k_2, k_3.

The phase of the state appearing in $\prod_{j=1}^{\infty}\{a^{h_j}(\boldsymbol{k}_j)/j\}\psi(0(\boldsymbol{k}))$ is infinite. Specially, we can also construct the state $i^{\infty}\psi(1(\boldsymbol{k}))$ which is the component of the state $\overline{\exp i\varphi((2\pi)^{3/2}\delta_{\boldsymbol{x}=0,0,0)})}\psi(0(\boldsymbol{k}))$. From Examples 2 and 3 we see the following two facts:

(1) the various relations (about to singularity) between the operator valued distribution $\varphi(f)$ and the singular function $\varphi(\boldsymbol{x})$ through a sort exponential function.

(2) the possibility to be related to the field with interaction by $\exp i\varphi(f)$ using the generalized singular testing function f.

References

[1] H. Yamagata: The role of mollifiers in Wightman functions. Proc. Japan Acad., 200 (1964).
[2] T. Ishihara and H. Yamagata: On distributions and spaces of sequences. I–IV. Proc. Japan Acad., **38** (1961).
[3] J. Von Neumann: On infinite products. Comp. Math., **46**, 1 (1939).
[4] R. Haag: Canonical commutation relations in Field theory and functional integration. Lectures in Theoretical Physics, Vol. III, Interscience publishers, New York (1960).
[5] L. Schwartz: Théorie des distribution. I, II. Hermann (1950, 1951).