72. *Product of Minimal Topological Spaces*

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A topological space $(X, \mathfrak{T})$ is said to be minimal Hausdorff if $\mathfrak{T}$ is Hausdorff and there exists no Hausdorff topology on X strictly weaker than $\mathfrak{T}$. In the same way, a topological space $(X, \mathfrak{T})$ is said to be minimal regular if $\mathfrak{T}$ is regular and there exists no regular topology on X strictly weaker than $\mathfrak{T}$. These definitions are due to M. P. Berri and R. H. Sorgenfrey ([1], [2]).

It is the purpose of this note to give an affirmative answer to the question, "Is the topological product of minimal Hausdorff spaces necessarily minimal Hausdorff ?", which is one of the problems which are not yet solved in the paper [1]. The converse of this question is already proved in [1], namely, if the nonempty product space is minimal Hausdorff, then each factor space is minimal Hausdorff.

We shall next obtain some results concerning minimal regular spaces.

§1. Product of minimal Hausdorff spaces. The following facts which are concerned with minimal Hausdorff spaces have been shown in [1].

A necessary and sufficient condition that a Hausdorff space $(X, \mathfrak{T})$ be minimal Hausdorff is that $\mathfrak{T}$ satisfies property S:

$S(1)$ Every open filter-base (which composed exclusively of open sets) has an adherent point:

$S(2)$ If an open filter-base has a unique adherent point, then it converges to this point.

A Hausdorff space X which satisfies $S(2)$ also satisfies $S(1)$.

From [3, p. 110], property $S(1)$ is a necessary and sufficient condition for a Hausdorff space to be absolutely closed.

Theorem 1. *The topological product of minimal Hausdorff spaces is minimal Hausdorff.*

Proof. Let $Z=\prod_{\alpha} X_\alpha$, where X_α is a minimal Hausdorff space for all α. Let $\mathfrak{F}$ be an open filter-base with a unique adherent point $z^0=(x_\alpha^0)\in Z$. We shall show that $\mathfrak{F}$ converges to the point z^0. In order to show this, we divide the proof into three parts.

1) For every α, let $\mathfrak{F}_\alpha=\{F_\alpha=\mathrm{Pro}_\alpha F \mid F\in\mathfrak{F}\}$,*) then $\mathfrak{F}_\alpha$ is an open filter-base in X_α.

*) $\mathrm{Pro}_\alpha F$ denotes the projection of F into the factor space X_α.

It is obvious that $F_\alpha \in \mathfrak{F}_\alpha$ is open and for every $F_\alpha^1, F_\alpha^2 \in \mathfrak{F}_\alpha$, there exist $F^1, F^2 \in \mathfrak{F}$ such that $F_\alpha^1 = \mathrm{Pro}_\alpha F^1$ and $F_\alpha^2 = \mathrm{Pro}_\alpha F^2$. Since $\mathfrak{F}$ is a filter-base, there exists $F^3 \in \mathfrak{F}$ such that $F^3 \subset F^1 \frown F^2$. Let $F_\alpha^3 = \mathrm{Pro}_\alpha F^3$, then $F_\alpha^3 \subset F_\alpha^1 \frown F_\alpha^2$. Hence $\mathfrak{F}_\alpha$ is an open filter-base.

2) $\mathfrak{F}_\alpha$ has the unique adherent point x_α^0. Hence it converges to x_α^0.

Let α' be arbitrary and fixed and let $Z' = X_{\alpha'} \times \prod_{\alpha \neq \alpha'} X_\alpha$, then Z' is homeomorphic to Z. Hence we may assume that $Z = X_{\alpha'} \times \prod_{\alpha \neq \alpha'} X_\alpha$. Let $X = X_{\alpha'}$ and $Y = \prod_{\alpha \neq \alpha'} X_\alpha$, then $Z = X \times Y$. Let $\mathfrak{F}_X = \{F_X = \mathrm{Pro}_X F \mid F \in \mathfrak{F}\}$ and $z^0 = (x^0, y^0)$. Since $z^0 \in \overline{F}$, $x^0 \in \mathrm{Pro}_X \overline{F} \subset \overline{F}_X$. Hence x^0 is an adherent point of $\mathfrak{F}_X$. Let x be an arbitrary point such that $x \neq x^0$. We shall show that x is not an adherent point of $\mathfrak{F}_X$. Let $z = (x, y)$, where $y \in Y$, then z is not an adherent point of $\mathfrak{F}$, since $z \neq z^0$. Consequently there exist some open neighborhood $W_y(z) = V_y(x) \times U(y)$ of z and some $F^y \in \mathfrak{F}$ such that $V_y(x) \times U(y) \frown F^y = \phi$. On the other hand, $Y = \bigcup_{y \in Y} U(y)$. However, since Y is the product space of absolutely closed spaces, Y is absolutely closed [4, p. 138], then we have $Y = \bigcup_{i=1}^{n} \overline{U(y_i)}$. Since $V_{y_i}(x) \times U(y_i) \frown F^{y_i} = \phi$ and F^{y_i} is open, $V_{y_i}(x) \times \overline{U(y_i)} \frown F^{y_i} = \phi$. Let $V(x) = \bigcap_{i=1}^{n} V_{y_i}(x)$ and let $F^0 \in \mathfrak{F}$ be an element which is contained in $\bigcap_{i=1}^{n} F^{y_i}$. Then $\phi = \bigcup_{i=1}^{n} \{V_{y_i}(x) \times \overline{U(y_i)} \frown F^{y_i}\} \supset \bigcup_{i=1}^{n} \{V(x) \times \overline{U(y_i)}\} \frown F^0 = V(x) \times Y \frown F^0$. Therefore x is not an adherent point of $\mathfrak{F}_X$.

3) F converges to z^0.

Let $W(z^0) = \prod_\alpha V(x_\alpha^0)$ be a neighborhood of z^0, where $V(x_\alpha^0)$ is a neighborhood of x_α^0 for $\alpha = \alpha_i$ $(i = 1, 2, \cdots, n)$ and $V(x_\alpha^0) = X_\alpha$ for $\alpha \neq \alpha_i$. For α_i, there exists $F_{\alpha_i}^i \in \mathfrak{F}_{\alpha_i}$ such that $F_{\alpha_i}^i \subset V(x_{\alpha_i}^0)$, since $\mathfrak{F}_{\alpha_i}$ converges to $x_{\alpha_i}^0$. Hence $\mathrm{Pro}^{-1} V(x_{\alpha_i}^0) \supset \mathrm{Pro}^{-1} F_{\alpha_i}^i \supset F^i \in \mathfrak{F}$. Take some $F \in \mathfrak{F}$ such that $F \subset \bigcap_{i=1}^{n} F^i$. Since $W(z^0) = \prod_\alpha V(x_\alpha^0) = \bigcap_{i=1}^{n} \mathrm{Pro}^{-1} V(x_{\alpha_i}^0) \supset \bigcap_{i=1}^{n} F^i \supset F$, $\mathfrak{F}$ converges to x.

Therefore Z is a minimal Hausdorff space. This completes the proof.

§2. **Product of minimal regular spaces.** An open filter-base will be called regular if it is equivalent to a closed filter-base (cf. [2]).

The following fact has been shown by M. P. Berri and R. H. Sorgenfrey [2].

In order that a regular space be minimal regular, it is necessary and sufficient that every regular filter-base having a unique adherent point is convergent.

Theorem 2. *The product of a minimal regular space and a compact Hausdorff space is minimal regular.*

Proof. Let $Z = X \times Y$, where X is minimal regular and Y is compact Hausdorff. Let $\mathfrak{F}$ be a regular filter-base having a unique adherent point $z^0=(x^0, y^0)$. Let $\mathfrak{H}$ be a closed filter-base equivalent to $\mathfrak{F}$. And let $\mathfrak{F}_X=\{F_X=\mathrm{Pro}_X F \mid F\in\mathfrak{F}\}$ and $\mathfrak{H}_X=\{H_X=\mathrm{Pro}_X H \mid H\in\mathfrak{H}\}$. $\mathfrak{F}_X$ is an open filter-base and since Y is compact, $\mathfrak{H}_X$ is a closed filter-base equivalent to $\mathfrak{F}_X$. Hence $\mathfrak{F}_X$ is a regular filter-base in X.

In the same way as 2) of Theorem 1, we can prove that $\mathfrak{F}_X$ has a unique adherent point x^0, hence $\mathfrak{F}_X$ converges to x^0.

We shall show that $\mathfrak{F}$ converges to z^0. Let $V(x^0)\times U(y^0)$ be an arbitrary neighborhood of z^0. For all $y\notin U(y^0)$, let $z=(x^0, y)$, then z is not an adherent point of $\mathfrak{F}$. Thus there exist $V_y(x^0)\times U(y)$ and $F^y\in\mathfrak{F}$ such that $V_y(x^0)\times U(y)\cap F^y=\phi$. Since $\bigcup_{y\notin U(y^0)} U(y)\cup U(y^0)=Y$ and Y is compact, $\bigcup_{i=1}^{n} U(y_i)\cup U(y^0)=Y$. Let $V_0(x^0)=V(x^0)\cap(\bigcap_{i=1}^{n} V_{y_i}(x^0))$. Since $\mathfrak{F}_X$ converges to x^0, there exists $F_X^0\in\mathfrak{F}_X$ such that $V_0(x^0)\supset F_X^0$. Then $F^0\subset V_0(x^0)\times Y$ and $F^{y_i}\cap F^0\subset V_0(x^0)\times U(y_i)^c$, where $U(y_i)^c$ denotes the complement of $U(y_i)$. Then $\bigcap_{i=1}^{n} F^{y_i}\cap F^0\subset\bigcap_{i=1}^{n}\{V_0(x^0)\times U(y_i)^c\}\subset V_0(x^0)\times U(y^0)\subset V(x^0)\times U(y^0)$. Now, let $F\in\mathfrak{F}$ be such that $F\subset\bigcap_{i=1}^{n} F^{y_i}\cap F^0$, then $F\subset V(x^0)\times U(y^0)$, so that $\mathfrak{F}$ converges to z^0. This completes the proof.

Remark. In the same way as [1, Theorem 1.10], we can prove that if a nonempty product space is minimal regular, then each factor space is minimal regular.

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References

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