69. *On the Explanation of Observables and States*

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§1. **Introduction.** In the previous paper [1], three kinds of multiplications of operator valued functions are given. But the differences among them are very delicate and important. Here, giving the exact definitions of the testing functions and mollifiers, the differences among these multiplications are discussed.

Since the multiplications used in axiomatic relativistic quantum field theory is (2) in [1], the non local field appears [7]. Here for the purpose of the construction of local observables appearing in Wightman function, the functional integration is used [6].

On the other hand Von Neumann has constructed the direct product space to represent the state vectors. But, between this and true state vectors' space there are following differences [2]:

(1) True space of state vectors is not a Hilbert space but a space consisting of vectors with unit length.

(2) In Von Neumann's direct product space, the treatment of the states with infinite phase amplitude is not necessarily faithful to the treatment of the state vectors.

Hence, in this paper, by considering the formal meaning of vectors contained in Von Neumann's direct product space, the useful new state is constructed by the Gelfand construction in [3]. Furthermore, the true character of this constructed states is shown.

§2. **The relation among the three kinds of multiplications.** Let's also use the most of the notations and definitions found in [1], [3].

In [1], the three kinds of multiplications of the operator valued functions have been defined, and the differences among them also have been investigated. Here, let's give the deeper consideration to the differences among them.

For these definitions in [1], the mollifiers, the testing functions or both of them are always used.

Suppose that $\varphi(x)$ is represented by the triplet $[\varphi(x), \{\rho(x)\}, \{f(x)\}]$ and the multiplications are defined in the set of these triplets. Here $\varphi(x)$ is an operator valued function, $\{\rho(x)\}$ is the set of mollifiers and $\{f(x)\}$ is the set of testing functions.

For these definitions of multiplications $\{\rho(x)\}$ and $\{f(x)\}$ are not necessarily used at the same time.

Namely in the multiplication of (1) in [1] $\varphi(x)$ and $\{f(x)\}$ are used. In (2) (in [1]) $\varphi(x)$ and $\{\rho(x)\}$ are used. And in (3) (in [1]) $\varphi(x)$, $\{\rho(x)\}$, and $\{f(x)\}$ are used. To obtain the more precise and delicate relations among these multiplications are our aim. Next we show this. For this purpose the difference between the mollifiers and the testing functions which is not yet clear is necessary. These physical meanings are the following: if the mollifiers are essentially used, this multiplication is constructed by using non local fields, and if the testing functions are essentially used, this multiplication is constructed by using local fields. In this paper, the following explanation is used to show this difference between the mollifiers and the set of testing functions [4], [5].

Definition 1. 1) *Let's consider the smooth function* $\varphi(x)$ *with the carrier not to be one point. If* $\varphi(x)$ *is essentially used to construct the product, or if it is used to construct the multiplicator and the multiplicand, we say that* $\varphi(x)$ *is a mollifier.* 2) *If sufficiently many* $\varphi(x)$*'s are used to represent the product after the construction of it, we say that* $\varphi(x)$*'s are testing functions.*

Using Definition 1, let's show the deeper and more delicate difference among the multiplications defined in [1].

A) The difference between the multiplication of (1) in [1] and (2) in [1]: Since in the multiplication of (2)

$$\varphi\circ\psi(f)=\varphi(f)\cdot\psi(f)=\langle\varphi(x)\times\psi(x'),\ f(x)\times f(x')\rangle$$

the set of functions $\{f(x)\times f(x')\}$ does not contained sufficiently many elements to represent the product $\varphi(x)\times\psi(x')$ and since $f(x)$ is rather used to construct the multiplicator and the multiplicant, it follows that $f(x)\times f(x')$ is not used as the testing functions but used as the set of the mollifiers.

In (1) $h(x, x')$ or $f(x)g(x')$ is evidently the set of testing function. Then, there is a difference between the multiplication of (1) and of (2).

(1) uses the local fields and (2) uses the non local field.

Afterwards, we will try to construct the operators corresponding to $\exp i\varphi(f)$ using the multiplication which has the characters of both (1) and (3).

B) Between (2) in [1] and (3) in [1]. Let's consider the multiplication of (3) defined in [1]. Namely $\varphi\circledcirc\psi(f)=[\{\langle\varphi*\rho_\varepsilon\cdot\psi*\rho_{\varepsilon'},\ f\rangle\}]$ for $f(x)\in(D(R^3))$, where $\rho_\varepsilon(x), \rho_{\varepsilon'}(x)\in(D(R^3))$, $\lim_{\varepsilon\to0}\rho_\varepsilon(x)=\delta$ and $\lim_{\varepsilon'\to0}\rho_{\varepsilon'}(x)=\delta$ in (D') and $\lim_{\varepsilon\to0}|x|^m\rho_\varepsilon(x)=0$ and $\lim_{\varepsilon'\to0}|x|^m\rho_{\varepsilon'}(x)=0$ uniformly (m is an arbitrary fixed integer).

In this product, f is evidently a testing function. $\rho_\varepsilon(x)$ and $\rho_\varepsilon(x')$ has the characters of mollifier, but from the conditions $\lim_{\varepsilon\to0}\rho_\varepsilon(x)=\delta$ and $\lim_{\varepsilon'\to0}\rho_{\varepsilon'}(x)=\delta$, the carrier of the sequence $\{\rho_\varepsilon(x)\}$ and the carrier of the sequence $\{\rho_{\varepsilon'}(\boldsymbol{x})\}$ are essentially one point.

Then $\{\rho_{\varepsilon}(\boldsymbol{x})\}$ and $\{\rho_{\varepsilon'}(\boldsymbol{x})\}$ are not simply the mollifier, but the tool for the construction of the sequences such that each of them approach to $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$.

Hence $\{\rho_{\varepsilon}(\boldsymbol{x})\}$ etc. are essentially different from f used in the multiplication of (2).

C) Between (3) in [1] and (1) in [1]. We can rewrite the multiplication of (3) defined in [1] to the form $\varphi \circledcirc \psi(f) = [\{\langle\langle\varphi(\tilde{\boldsymbol{x}}) \times \psi(\tilde{\boldsymbol{x}}'), \rho_{\varepsilon}(\tilde{\boldsymbol{x}}-\boldsymbol{x}) \times \rho_{\varepsilon'}(\tilde{\boldsymbol{x}}'-\boldsymbol{x})\rangle, f(\boldsymbol{x})\rangle\}]$.

But the conditions $\lim_{\varepsilon\to 0} \rho_{\varepsilon}(\boldsymbol{x}) = \delta$ and $\lim_{\varepsilon'\to 0} \rho_{\varepsilon'}(\boldsymbol{x}) = \delta$ construct the difference between (3) and (1).

Namely $\rho_{\varepsilon}(\tilde{\boldsymbol{x}}-\boldsymbol{x}) \times \rho_{\varepsilon'}(\tilde{\boldsymbol{x}}'-\boldsymbol{x})$ is not essentially a testing function depending to $\boldsymbol{x}$ but is used to construct the sequence converging to $\varphi(\tilde{\boldsymbol{x}}) \times \psi(\tilde{\boldsymbol{x}})$, because the carrier of $\{\rho_{\varepsilon}(\tilde{\boldsymbol{x}}-\boldsymbol{x}) \times \rho_{\varepsilon'}(\tilde{\boldsymbol{x}}'-\boldsymbol{x})\}$ is essentially one point for fixed $\boldsymbol{x}$.

Now let's construct the local observable appearing in Wightman function, using the conception of the multiplication of (1) and (3) [1].

Let's construct the formal infinite direct product of $C_0^\infty(\boldsymbol{x}_n)$. And let $\prod_n \otimes C_0^\infty(\boldsymbol{x}_n)$ denote this space. Let $A(\prod_n \otimes C_0^\infty(\boldsymbol{x}_n))$ denote the linear aggregate of this space. We use $A(\prod_n \otimes C_0^\infty(\boldsymbol{x}_n))$ because $\prod_n \otimes C_0^\infty(\boldsymbol{x}_n)$ is too small to use.

Definition 2. *Let* $([\exp(i\varphi)]*(\delta))_{\boldsymbol{x}_i=0} \prod_k \otimes \psi(0(\boldsymbol{k}))$ *denote the set of sequences* $[\{\langle[1\times 1\times\cdots + i\varphi\times 1\times\cdots + i\varphi\times i\varphi/2\times 1\times\cdots+\cdots]* f_n(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots)\rangle \cdot \prod_k \otimes \psi(0(\boldsymbol{k}))\}]$, *for* $f_n(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots)$ *contained in* $A(\prod_n \otimes C_0^\infty(\boldsymbol{x}_n))$.

Here, the sequence $\{f_n(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots)\}$ *satisfies the following conditions*:

a) $f_n(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots)$ *can be decomposed in the form* $f_n(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots) = \sum_{m=1}^{Mn} C_{mn} \varphi_{mn}^{(1)}(\boldsymbol{x}_1) \varphi_{mn}^{(2)}(\boldsymbol{x}_2) \cdots \varphi_{mn}^{(k)}(\boldsymbol{x}_k) \cdots$.

b) $\sum_{m=1}^{Mn} C_{mn} = 1, \int \varphi_{mn}^{(k)}(\boldsymbol{x}_k)\, d\boldsymbol{x}_k = 1$.

c) $U_{m=1}^{Mn}$ *{carrier* $(\varphi_{mn}^{(k)}(\boldsymbol{x}_k))\}$ *tend to one point* $\boldsymbol{x}_k = 0$ *for any* k *as* $n \to \infty$. *Namely* $[\{[(1\times 1\times\cdots + i\varphi\times 1\times\cdots + i\varphi\times i\varphi/2\times 1\times\cdots+\cdots)*(\sum_{m=1}^{Mn} C_{mn} \varphi_{mn}^{(1)}(\boldsymbol{x}_1) \varphi_{mn}^{(2)}(\boldsymbol{x}_2) \cdots)]_{x_i=0} \cdot \prod_k \otimes \psi(0(\boldsymbol{k}))\}] = \Big[\Big\{\Big[\sum_{m=1}^{Mn} C_{mn} \int \varphi_{mn}^{(1)}(\boldsymbol{x}_1)\, d\boldsymbol{x}_1 \times \times \int \varphi_{mn}^{(2)}(\boldsymbol{x}_2)\, d\boldsymbol{x}_2 \cdots + \sum_{m=1}^{Mn} C_{mn} (i\varphi(\boldsymbol{x}_1) * \varphi_{mn}^{(1)}(\boldsymbol{x}_1)) \int \varphi_{mn}^{(2)}(\boldsymbol{x}_2)\, d\boldsymbol{x}_2 \cdots + \cdots\Big]_{x_i=0} \cdot \prod_k \otimes \psi(0(\boldsymbol{k}))\Big\}\Big]$.

Using this definition, the local observables can be defined. This situation is similar to one such that non linear equation can be linearlized using the functional integration. To construct the states which is not contained in the domain of free hamiltonian, we have used the above $\{f_n(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots)\}$ instead of the testing functions. For the states constructed from the above local observable, we give two explanation; one is that according to Von Neumann, and the other is that according to the following Definition 3. The use of the sequence $\{f_n(x_1, x_2, \cdots)\}$ is to give the rule of conditional convergence. Namely, $\delta_{x_1}\times\delta_{x_2}\times\cdots$ really contains the infinitely many $[\{f_n(x_1, x_2, \cdots)\}]$. And it corresponds to the following cut-off:

$$\begin{aligned} i\varphi(x_1)\times i\varphi(x_2)/2\times i\varphi(x_3)/3\times\cdots * \textstyle\sum_{m=1}^{Mn} C_{mn}\varphi_{mn}^{(1)}(x_1)\varphi_{mn}^{(2)}(x_2)\varphi_{mn}^{(3)}(x_3)\cdots \\ =\textstyle\sum_{m=1}^{Mn} C_{mn}(i\varphi(x_1)*\varphi_{mn}^{(1)}(x_1))\cdot(i\varphi(x_2)/2*\varphi_{mn}^{(2)}(x_2))\cdots \\ =\textstyle\sum_{m=1}^{Mn} C_{mn}\mathfrak{F}^{-1}(i\mathfrak{F}\varphi(x_1)\cdot\mathfrak{F}\varphi_{mn}^{(1)}(x_1))\cdot\mathfrak{F}^{-1}(i\mathfrak{F}\varphi(x_2)/2\cdot\mathfrak{F}\varphi_{mn}^{(2)}(x_2))\cdots. \end{aligned}$$

And $\lim_{n\to\infty}\sum_{m=1}^{Mn} C_{mn}\mathfrak{F}\varphi_{mn}^{(1)}(x_1)\cdot\mathfrak{F}\varphi_{mn}^{(2)}(x_2)\cdots\mathfrak{F}\varphi_{mn}^{(l)}(x_l)\cdots=1(k_1, k_2, \cdots)$.

Then the conditional convergence in [3] and in the following §3 can be represented by using this form.

§3. The construction of the states. We denote by $\psi(n_k)$ the eigenfunction with norm 1 corresponding to n particle with momenta $\boldsymbol{k}$.

Let $a^h(\boldsymbol{k})$ denote the unbounded operator $a(\boldsymbol{k})$ for $h=0$, and denote $a^+(\boldsymbol{k})$ for $h=1$.

Let $b(a^h(\boldsymbol{k}))$ denote the coefficient of $a^h(\boldsymbol{k})$ in $\varphi(f)$.

Let $\exp(i\theta^h(\boldsymbol{k}))$ denote the complex number $b(a^h(\boldsymbol{k}))/|b(a^h(\boldsymbol{k}))|$, whose absolute value is 1.

Definition 3. We denote by $[\exp(i\varphi(f))]\cdot\prod_k\otimes\psi(n_k)$ the formal sum of the following state vectors:

(1) $\{\prod_{j=1}^{n}(b(a^{h_j}(k_j))a^{h_j}(k_j)dk_j)/j\}\cdot\prod_k\otimes\psi(n_k)$ where $h_j=0$ or 1.

(2) i) Let σ denote the set of all possible operators $\prod_{j=1}^{\infty}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\}$ such that the state $\prod_k\otimes\psi(n_k)$ can be constructed from $\prod_k\otimes\psi(0_k)$ by using the creation and annihilation operators in this infinite product.

ii) Let σ_N denote the set of all operators $\prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\}$ constructed from $\prod_{j=1}^{\infty}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\}$ in σ.

iii) Let $\widetilde{C}(n(\boldsymbol{k}), \prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\})$ denote the complex number $\prod_{j=1}^{N}\{|b(a^{h_j}(\boldsymbol{k}_j))|/j\}\cdot\|\prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)\prod_k\otimes\psi(0_k)\|$.

If $C(n(\boldsymbol{k}), \theta(\boldsymbol{k}))=\lim_{N\to\infty}\widetilde{C}(n(\boldsymbol{k}), \prod_{j=1}^{N}\{a^{h_j}(\boldsymbol{k}_j)d\boldsymbol{k}_j/j\})$ is definite and non-zero, we say that the state $C(n(\boldsymbol{k}), \theta(\boldsymbol{k}))\prod_k\otimes e^{i\theta^h(\boldsymbol{k})}\psi(n_k)$ is also the component of the state $[\exp(i\varphi(f))]\cdot\prod_k\otimes\psi(0_k)$, where $\theta^h(\boldsymbol{k})$ is the sum of amplitudes related to the momenta $\boldsymbol{k}$. The above Definition 3 gives the new explanation to some states with infinite phase amplitude which cannot be represented by $\overline{\exp(i\varphi(f))}\cdot\prod_k\otimes\psi(0_k)$ found in [3].

Using Definition 2 corresponding to Definition 3, let's construct the state $\prod_k\otimes e^{i\frac{\pi}{2}}\psi(1_k)$ which is equal to 0 in Von Neumann's infinite direct product space.

Example. Let's enclose the considered system in a box of finite volume V. Choose the generalized function $\sqrt{V}\delta$ as the function f.

Then we can obtain the following formula formally,

$$\exp i\varphi(f)=\sum_{n=0}^{\infty}(1/n!)[i\sum_{k=(k_1,k_2,k_3)}(a^+(k)+a(k))]^n$$

for non negative integer k_1, k_2, k_3. Using the conditional convergence defined in [3] or the corresponding selection of the sequence $\{f_n(x_1, x_2, \cdots)\}$, the component

$$\prod_k \otimes e^{i\frac{\pi}{2}}\varphi(1_k) \quad \text{of} \quad ([\exp(i\varphi)]*(\delta))_{x_k=0}\prod_k\otimes\varphi(0_k)$$

is obtained by the explanation in Definition 3.

Here we show this conditional convergence defined in [3] once more.

Ordering the set of triplet (k_1, k_2, k_3), construct the sequence $(0, 0, 0)$ $(1, 0, 0)$ $(0, 1, 0)$ $(0, 0, 1)$ $(2, 0, 0)\cdots$. Using the first m terms of the above sequence, construct the $m!$ sequences. Operate to the state $\prod_k\otimes\psi(0_k)$ m creation operators $ia^+(k)$ corresponding to the terms of the above some sequence. Since $m!/m!=1$, it will be seen that the sum of all this states becomes to $\prod_k\otimes e^{i\frac{\pi}{2}}\psi(1_k)$ as m tend to ∞.

The inner product between those states are the following:

$$\langle\prod_k\otimes\psi_k, \prod_k\otimes\varphi_k\rangle=\prod_k\langle\psi_k, \varphi_k\rangle.$$

This inner product is not always number but formal infinite product of complex numbers. We only require that for fixed k the finite inner product $\langle\psi_k, \varphi_k\rangle$ is defined. Here, the linearlity has only the probabilistic meaning.

At last, let's show the exact meaning of the component $\prod_{j=1}^{\infty}\{b(a^h(k_j))\cdot a^h(k_j)dk_j/j\}$ appearing in Definition of [3] and in Definition 3 of this paper.

Let's decompose $\varphi(f_n)$ in the following form: $\varphi(f_n)=\int\lambda dE_{\varphi(f_n)}(\lambda)$, where $\lim_{n\to\infty}f_n=\delta$. On the other hand, we can represent $\varphi(f_n)$ by the following form $\varphi(f_n)=\sum_k(C_n(k)a^+(k)+\tilde{C}_n(k)a(k))$, where $C_n(k)=0$ $\tilde{C}_n(k)=0$ for sufficiently large $|k|$ and $\lim_{n\to\infty}C_n(k)=1$ for all k.

Since $||\varphi(f_n)\Phi||\geqq\sqrt{\sum_k|C_n(k)|^2}\ ||\Phi||$ for all states Φ, then the spectral measure $dE_{\varphi(f_n)}(\lambda)$ is distributed in the domain $|\lambda|\geqq\sqrt{\sum_k|C_n(k)|^2}$.

Since $\lim_{n\to\infty}\sqrt{\sum_k|C_n(k)|^2}=\infty$, it is obvious that $\varphi(\delta)$ is related to $\lambda=\pm\infty$.

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