100. *Markovian Systems of Measures on Function Spaces*

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A Markovian process defined on a path space is a system of non-negative probability measures on a function space. In this note we construct systems of signed measures corresponding to contraction semigroups (Theorem 1). These systems can be regarded as a generalization of Markovian processes. It is well known that the generator of a continuous Markovian process on a Euclid space is a generalized elliptic differential operator of second order. An analogous result holds also in our cases (Theorem 2).

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1. Let (E, ρ) be a σ-compact metric space, and $\boldsymbol{C}$ be the Banach space consisting of all real-valued, continuous, and bounded functions on E with the uniform norm $\|\cdot\|$. Let T_t be a strongly continuous contraction semigroup on $\boldsymbol{C}$. We assume that the operators T_t are expressed in the integral form:

$$T_t f(x)=\int_E f(y)P(t, x, dy) \quad (f \in \boldsymbol{C}),$$

where $P(t, x, \cdot)$ are signed measures[1] which satisfy the Kolmogoroff-Chapman equation

$$P(t+s, x, \cdot)=\int_E P(t, x, dy)P(s, y, \cdot).$$

Let ∂ be an extra point added to E and put

$$\widetilde{P}(t, x, \cdot)=\begin{cases} P(t, x, E\frown\cdot)+\delta_\partial(\cdot)\{1-P(t, x, E)\}, & \text{if } x\in E, \\ \delta_\partial(\cdot) & \text{, if } x=\partial, \end{cases}$$

where δ_∂ is the Dirac measure. Then $\widetilde{P}(t, x, \cdot)$ are measures on $E\smile\partial$, which satisfy the Kolmogoroff-Chapman equation and also the equality

$$\text{(1)} \qquad \widetilde{P}(t, x, E\smile\partial)=1.$$

We assume in the following that the function 1 belongs to the domain $\mathfrak{D}(\mathcal{G})$ of the generator $\mathcal{G}$ of the semigroup T_t. We have

$$\text{(2)} \qquad |\widetilde{P}|(t, x, E\smile\partial)\leqq e^{\gamma t},\text{[2]}$$

where $\gamma=\|\mathcal{G}1\|$.

Let Ω be the set of all functions that are defined on $[0, \infty)$ and take values from $E\smile\partial$. We write

1) Hereafter we omit the adjective "signed".

2) By $|\widetilde{P}|$ we denote the total variation of $\widetilde{P}$.

$$p_t(\omega)=\omega(t) \text{ for } \omega\in\Omega \text{ and } t\in[0, \infty).$$

We denote by $\mathfrak{F}_T$ the algebra generated by the sets of the form

$$\mathrm{I}=\{\omega: p_{t_1}(\omega)\in\Gamma_1,\ p_{t_2}(\omega)\in\Gamma_2,\cdots, p_{t_n}(\omega)\in\Gamma_n\},$$

where $0\leqq t_1<t_2<\cdots<t_n\leqq T$ and $\Gamma_1, \Gamma_2,\cdots,\Gamma_n$ are Borel sets in $E\smile\partial$. Put

$$\mathrm{P}_x(\mathrm{I})=\int_{\Gamma_1}\int_{\Gamma_2}\cdots\int_{\Gamma_n}\widetilde{P}(t_1, x, dy_1)\widetilde{P}(t_2-t_1, y_1, dy_2)\cdots\widetilde{P}(t_n-t_{n-1}, y_{n-1}, dy_n).$$

Because of the equation (1) and of the Kolmogoroff-Chapman equation the value $\mathrm{P}_x(\mathrm{I})$ is independent of the expression of I. The set function $\mathrm{P}_x(\cdot)$ is easily extended to a finitely additive measure on $\mathfrak{F}_T$, the total variation of which is bounded by e^{rT} because of the inequality (2). Therefore P_x is decomposed into the positive part and the negative one (Jordan's decomposition). By Kolmogoroff's extension theorem [3] both parts are extended to σ-additive measures on the σ-algebra $\mathfrak{B}_T$ generated by $\mathfrak{F}_T$. We need the σ-compactness of the space E for application of the theorem. Thus P_x is extended to $\mathfrak{B}_T$.

An elementary calculation shows that for $0<t<u<T$,

$$(3)\qquad \mathrm{P}_x(\mathfrak{B}_T)[\omega: \rho(p_t(\omega), p_u(\omega))>\varepsilon]$$
$$\leqq e^{rT}\boldsymbol{M}_x(\mathfrak{B}_T)[\widetilde{P}\{u-t, p_t(\omega), U(p_t(\omega), \varepsilon)^c\}+e^{r(u-t)}-1].$$[3]

From this inequality and the strong continuity of the semigroup T_t we get a convergence

$$\mathrm{P}_x(\mathfrak{B}_T)[\omega: \rho(p_t(\omega), p_u(\omega))>\varepsilon]\to 0 \quad (u-t\downarrow 0),$$

which enables us to apply Slutsky's method [4] in order to obtain the following

Proposition 1. *There exists a mapping $x_t(\omega)$ measurable in (t, ω) such that*

$$\mathrm{P}_x(\mathfrak{B}_T)[\omega: x_t(\omega)\neq p_t(\omega)]=0, \quad \text{for} \quad t\leqq T.$$

The objects which we have constructed satisfy the following conditions.

a) $x(t, \omega)\equiv x_t(\omega)$ is a measurable mapping from $[0, \infty)\times\Omega$ into $E\smile\partial$.

b) $\mathfrak{B}_t$ is a σ-algebra of subsets of Ω for $t>0$.

c) $\mathfrak{B}_t\supset\mathfrak{B}_s$ for $t>s$.

d) $\mathrm{P}_x(\cdot)$ is a measure on $\mathfrak{B}_t$.

e) $\{\omega: x_t(\omega)\in\Gamma\}\in\mathfrak{B}_t$ for any Borel set Γ in E.

f) $\mathrm{P}_x[x_t\in\Gamma]$ is measurable in x.

g) $\mathrm{P}_x(\mathfrak{B}_T)[x_0(\omega)\neq x]=0$ for $T>0$.

h) $\mathrm{P}_x[\mathrm{I}_1\theta_t\mathrm{I}_2]=\int_{\mathrm{I}_1}\mathrm{P}_{x_t(\omega)}(\mathrm{I}_2)\mathrm{P}_x(d\omega)$ for any $\mathrm{I}_1\in\mathfrak{B}_t$ and $\mathrm{I}_2\in\mathfrak{B}\equiv\bigcup_{t>0}\mathfrak{B}_t$.[4]

3) We denote by $\mathrm{P}_x(\mathfrak{B}_T)[A]$ the variation of P_x on A over the σ-algebra $\mathfrak{B}_T$, by $\boldsymbol{M}_x(\mathfrak{B}_T)$ the integration by the measure $\mathrm{P}_x(\mathfrak{B}_T)[d\omega]$, by $U(x, \varepsilon)$ the ε-neighbourhood of x.

4) The definition of the operator θ_t is found in [1].

We call $X \equiv (x_t, \mathfrak{B}_t, \mathrm{P}_x, \theta_t)$ *a Markovian system on the phase space* E *defined on the space* Ω *of elementary events.*

Theorem 1. *Let be given a* σ*-compact metric space* (E, ρ) *and a strongly continuous contraction semigroup* T_t *on the Banach space* $\boldsymbol{C}$ *consisting of all real-valued, continuous, bounded functions on* E. *Assume that the operators* T_t *are expressed in the integral form, and that the domain of the generator of the semigroup contains the function* 1.

Then there exists a Markovian system $X = (x_t, \mathfrak{B}_t, \mathrm{P}_x, \theta_t)$ *such that for any* $f \in \boldsymbol{C}$,

$$T_t f(x) = \boldsymbol{M}_x[f(x_t)].$$

2. Kac's theorem in the theory of Markovian processes is generalized to our cases.

Proposition 2. *Let* V *be an element of* $\boldsymbol{C}$. *Put, for* $f \in \boldsymbol{C}$,

$$T_t' f(x) = \boldsymbol{M}_x\left[f(x_t) \exp\left\{\int_0^t V(x_s)\, ds\right\}\right].$$

Then T_t' *is a strongly continuous semigroup on* $\boldsymbol{C}$, *the generator of which is* $\mathfrak{G} + V$.

The analogue of Kinney's estimate on the right-continuity of Markovian processes is valid.

Proposition 3. *We assume the completeness of the phase space* E. *If for any* $\varepsilon > 0$

$$\sup_{x \in E} \widetilde{P}\{t, x, U(x, \varepsilon)^c\} \to 0 \quad (t \downarrow 0),$$

then we can construct a Markovian system such that the measures P_x *concentrate on the set of* ω *for which* $x_t(\omega)$ *is right-continuous in* t.

The method for the construction is analogous to Theorem 6.3 of Dynkin [1]. Here we use a convergence

$$\sup_{x \in E} \mathrm{P}_x(\mathfrak{B}_T)[\rho(x_u, x_t) > \varepsilon] \to 0 \quad (u - t \downarrow 0),$$

which follows from the inequality (3) and the assumption of the proposition.

3. In the following we consider only continuous Markovian systems, i.e., Markovian systems whose measures P_x concentrate on the set of continuous paths. Such systems have the strongly Markovian property (Theorem 5.10 of Dynkin [1]). Dynkin's formula (Theorem 5.1 of [2]) is valid in our cases for *bounded* Markovian times.

For any neighbourhood U of x, we put

$$\tau_U(\omega) = S \wedge \inf\{t : x_t \notin U\},$$

where S is an arbitrarily fixed positive constant.

We call a point x an *irregular point*, if

$$\lim_{U \downarrow x} \frac{\boldsymbol{M}_x^-(\mathfrak{B}_T)[\tau_U]}{\boldsymbol{M}_x^+(\mathfrak{B}_T)[\tau_U]} = 1^{5)}$$

5) $\boldsymbol{M}_x^+(\mathfrak{B}_T)$ ($\boldsymbol{M}_x^-(\mathfrak{B}_T)$) is the integration by the positive (resp. negative) part of P_x.

for any $T \geqq S$. A Markovian system without any irregular point is called *regular*. The generator of a regular system is a restriction of the characteristic operator, i.e., there exists a sequence of neighbourhoods U_n of x converging to x, such that

$$\mathcal{G}g(x)=\lim_{n\to\infty}\frac{\boldsymbol{M}_x[g(x(\tau_{U_n}))]-g(x)}{\boldsymbol{M}_x[\tau_{U_n}]}$$

for any $g \in \mathfrak{D}(\mathcal{G})$.

When the phase space E is an interval (finite or infinite) on a line, we have a more concrete form of the characteristic operator.

Proposition 4. *Let X be a regular system on an interval E. Assume that there exists a local coordinate $\kappa_x \in \mathfrak{D}(\mathcal{G})$ with $\kappa_x^2 \in \mathfrak{D}(\mathcal{G})$ such that $\kappa_x(\cdot)=\cdot-x$ in a neighbourhood of x. Further assume that*

$$(\textit{diameter of } U)^2=O(\boldsymbol{M}_x^+(\mathfrak{B}_T)[\tau_U]), \quad U\downarrow x.$$

Then, for any twice continuously differentiable function $g \in \mathfrak{D}(\mathcal{G})$,

$$\mathcal{G}g(x)=a(x)g''(x)+b(x)g'(x)+c(x)g(x),$$

where $a(x)=\frac{1}{2}\mathcal{G}\kappa_x^2(x)$, $b(x)=\mathcal{G}\kappa_x(x)$ and $c(x)=\mathcal{G}1(x)$.

Theorem 2. *If in the previous proposition $a(x)$, $b(x)$, and $c(x)$ are all continuous, and if $[C^2(E)\backslash C^\infty(E)]\frown\mathfrak{D}(\mathcal{G})$ is not empty, then $a(x)$ is nonnegative.*

Proof. Without loss of generality, we may assume that $b(x)\equiv 0$. If our assertion is not true, there exist $l<r$ such that $a(x)<-\varepsilon<0$ for all $x\in[l, r]$. It is easy to see that $x_t^0(\omega)=x(t\wedge\tau(\omega), \omega)$ is a strongly Markovian system on the phase space $[l, r]$, where $\tau(\omega)=\inf\{t: x_t(\omega)\notin [l, r]\}$. By Volkonsky's random time change [5] corresponding to the additive functional

$$\alpha_t(\omega)=-\int_0^t a(x_s^0(\omega))\,ds,$$

we obtain a Markovian system (x_t^1, P_x) with the generator $-\dfrac{d^2}{dx^2}+c_1(x)$. Take g from $[C^2(E)\backslash C^\infty(E)]\frown\mathfrak{D}(\mathcal{G})$ and put

$$u(t, x)=\boldsymbol{M}_x\left[g(x_{T-t}^1)\exp\left\{-\int_0^{T-t}c_1(x_s^1)\,ds\right\}\right], \quad \text{for } t<T.$$

The function u is a solution of the initial-boundary value problem for the parabolic equation:

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}\\ u(0, x)=\boldsymbol{M}_x[g(x_T^1)],\\ u(t, l)=g(l),\ u(t, r)=g(r).\end{cases}$$

This problem, as is well known, has the unique solution, which is infinitely differentiable in x for any $t>0$. Therefore $u(T, x)=g(x)$ belongs to $C^\infty(E)$, which is incompatible with our situation that g is taken from $[C^2(E)\backslash C^\infty(E)]\frown\mathfrak{D}(\mathcal{G})$.

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