141. A Remark on Quasiideals of Regular Ring

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Let A be a ring. A is said to be regular (strongly regular) if for every element $a\in A$ there exists an element $x\in A$ such that $axa=a$ ($a=x^2a$). A subring Q of A is called a quasiideal of A if $AQ\cap QA \subseteq Q$. The concepts of a regular semigroup and quasiideal of a semigroup are described analogously. It is easy to see that a subring Q of a ring A is a quasiideal of A when and only when Q is a quasiideal of the multiplicative semigroup A. Various necessary and sufficient conditions for the regularity of rings are to be found in the literature (see [1], [3]). In this note we establish yet another such condition which does not appear to have been stated previously.

S. Lajos [2] has recently proved that the product of two quasiideals of a regular ring is a quasiideal. Thus, the collection of quasiideals of a regular ring forms a multiplicative semigroup. Considering this connection of regular ring with its quasiideals, we shall prove the following

Theorem. Let A be a ring and let $\mathfrak{A}$ be the collection of all quasiideals of A. Then A is regular if and only if $\mathfrak{A}$ is a regular multiplicative semigroup.

Before proving this theorem we shall recall the following result.

Lemma. A ring A is regular if and only if, for every quasiideal Q of A, $QAQ=Q$ (see [3]).

Proof of the theorem. Necessity. Assume that A is a regular ring and that $Q\in\mathfrak{A}$. Then, by the lemma, $Q=QAQ$, and $\mathfrak{A}$ is therefore a regular semigroup since $A\in\mathfrak{A}$.

Sufficiency. Suppose that $\mathfrak{A}$ is a regular semigroup. Let $Q\in\mathfrak{A}$. By virtue of the lemma we need only to show that $QAQ=Q$. In fact, by the regularity of $\mathfrak{A}$ there exists $P\in\mathfrak{A}$ such that $Q=QPQ$ and hence $Q\subseteq QAQ$. But, on the other hand, $QAQ\subseteq QA\cap AQ\subseteq Q$. Thus $QAQ=Q$ and the theorem follows.

It is known that a ring A is strongly regular if and only if every quasiideal of A is idempotent (see [1]). An immediate consequence is the following

Corollary. Let A be a ring and let $\mathfrak{A}$ be the collection of all quasiideals of A. Then A is strongly regular if and only if $\mathfrak{A}$ is an idempotent multiplicative semigroup.

Finally, it should be noted that by similar arguments we can show that a semigroup S is regular (strongly regular) if and only if the collection of all quasiideals of S forms a regular (idempotent) semigroup.

References

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