157. On the Cut-off Process in the Universal Hilbert Space

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1. Introduction. In what kind of linear (or Hilbert) spaces one should treat the Quantum field operator is an important problem both for practical calculation and for theoretical consideration. In the previous papers [1], [2] we investigate this problem. In contrast to the prevailing treatments where the endeavours to select one adequate irreducible representation space are done, we tried to find a way to use positively the universal Hilbert space [3] which may be considered to include all the inequivalent representation spaces provided that we could remove divergence.

In order to obtain the perturbative method in this space, the first principal problem is to introduce the adequate topology. In [2] we considered the topology which corresponds to the cut-off process. But these topologies in [2] are not enough general to be applied in the calculation of the quantum field theory. In this paper we introduce revised definitions of cut-off operators (§ 2). Using cut-off operators, topologies are introduced by two ways. In § 3 we consider the first way of introduction of topologies (direct method). In § 4 we consider the 2nd way of introduction of topology (dual space method). In [4], [5] Professor S. Kasahara discusses this problem and shows the necessary and sufficient conditions for the existence of topologies using dual space. We combine his method and the cut-off operations and obtain a practical method of limiting process using cut-off process.

2. Cut-off operators. Using the decomposition of the universal Hilbert space[1)] $\Pi\otimes\mathfrak{H}_\alpha=\Sigma_{\mathrm{c}}\oplus\Pi^{\mathrm{c}}\otimes\mathfrak{H}_\alpha$, we set a domain of cut-off operators by the following manner (1)$\sim$(3).

(1) We select a specific incomplete direct product $\Pi^{\mathrm{c}_0}\otimes\mathfrak{H}_\alpha=\Pi^{\mathrm{c}_0}$. Since the following theory proceeds isomorphic, we restrict our considerations to the case when the selected space is the closure of Fock space:[2)] $\Pi^F\otimes\mathfrak{H}_\alpha=\Pi^F=F$.

(2) Relating to the space F we classify the other incomplete direct products. The incomplete direct product Π^{c} in the universal Hilbert space is determined by any of normalized C_0-sequences $\{f_n^{\mathrm{c}}\}$ of

1) Terminology and notations not explained here are from J. von Neumann [6] and N. Bourbaki [10].

2) In [1], [2] we call this closure simply "Fock space".

c. We assign $\{f_n^c\}$ to every incomplete direct product Π^c and call $\Pi\otimes f_n^c$ *reprensentative* of Π^c. (For example we can assign every modified vacuum state as a representative of every irreducible representation space.) From $\Pi\otimes\mathfrak{H}_\alpha$ we exclude the incomplete direct product Π^q whose representative is orthogonal to F in the sense of quasi-convergent to 0. (This exclusion may be well since the vectors of two state in quantum field theory are considered to have definite relative phase.)

(3) Let $\{\Phi_\nu^c|\nu\}$ be a complete ortho-normal system of Π^c such that $\Phi_\nu^c=\Pi_\alpha\otimes\varphi_{\alpha i_\nu}^c(i_\mu\in I_\alpha^H)$[3] (c.f. Von Neumann [3] 4.1), which include representative i.e., $\Phi_0^c=\Pi_\alpha\otimes\varphi_{\alpha 0}^c$. We select an algebraic normalized base $B_\alpha=\{\tilde{\varphi}_{\alpha\lambda}|\lambda\in I_\alpha\}$[3] of the space $\mathfrak{H}_\alpha$ which contains $\{\varphi_{\alpha i}^c|i\in I_\alpha^H\}$. Let $[\Pi_\alpha\otimes B_\alpha^c]$ be a linear hull of $[\Pi_\alpha\otimes\tilde{\varphi}_{\alpha\lambda_\alpha}|\{\lambda_\alpha|\alpha\}$ runs through all the mapping $\{\alpha\}\rightarrow\{I_\alpha\}]$. We assign for each Π^c an algebraic linear space $[\Pi\otimes B_\alpha^c]$.

Example. We can assign $[\Pi\otimes B_\alpha^F]$ for the space F which contains the following states: (1) all the states $(n_1, n_2, \cdots)$[4] where n_i is an integer, and the $(p, p, p, \cdots)$ where p means a vector $f_\alpha=\sum_{n=0}^\infty c_n\varphi_{\alpha n}^F$[5] where $c_n=\sqrt{e^{-\lambda}\lambda^n/n!}$ and $\varphi_{\alpha n}^F$ corresponds to a particle number n.

Remark. Of course the space $[\Pi\otimes B_\alpha^F]$ can contain infinitely number of modified vacuum state. $H(\Gamma)$ in (c.f. [2], p. 24) is contained in closure of $[\Pi\otimes B_\alpha^F]$, but a vector such that $g=\sum_{i=1}^\infty c_i\Phi_i$ with $\sum_{i=1}^\infty|c_i|^2<\infty$, $\Phi_i=\Pi_{\alpha=1}^i\otimes\varphi_{\alpha 1}\otimes\Pi_{\alpha'=i+1}^\infty\otimes\varphi_{\alpha' 0}$, belongs to F and does not belong to $[\Pi\otimes B_\alpha^F]$. We see also that $\Pi_\alpha\otimes\frac{1}{\sqrt{2}}(\varphi_{\alpha 1}+\varphi_{\alpha 2})$ (c.f. [2], p. 25 Example 1) belongs to neither $\Pi\otimes B_\alpha^F$ nor $H(\Gamma)$ in this case.

At the 2nd place we define the two sort of cut-off operators P_N and $\hat{P}_N$ by the following (4)~(5).

(4) Let $\Phi\in\Pi'\otimes\mathfrak{H}_\alpha$ be a vector such that $\Phi=\sum_{i=1}^n c_{\vartheta_i}\Phi^{\vartheta_i}$ and $\Phi^{\vartheta_i}\in\Pi^{\vartheta_i}$, $\Phi^{\vartheta_i}=\sum_{\mu=1}^m c_\mu\Phi_\mu^{\vartheta_i}$ where $\Phi_\mu^{\vartheta_i}\in[\Pi\otimes B_\alpha^{\vartheta_i}]$. We introduce a cut-off operator using these unique expressions.

Definition 1. $P_{\vartheta N}$ is a linear mapping from $\Pi^\vartheta\frown[\Pi\otimes B_\alpha^\vartheta]$ into Π^F defined by the following equality:

For a fixed base vector $\Phi_\mu^\vartheta=\varphi_{1i_1}\otimes\cdots\otimes\varphi_{n_\mu i_{n_\mu}}\otimes\varphi_{n_\mu+1,0}^\vartheta\otimes\varphi_{n_\mu+2,0}^\vartheta\otimes\cdots$,

$$P_{\vartheta N}\Phi_\mu^\vartheta=\begin{cases}\varphi_{1i_1}\otimes\cdots\otimes\varphi_{n_\mu i_{n_\mu}}\otimes\psi_{n_\mu+1,0}^F\otimes\psi_{n_\mu+1,0}^F\otimes\cdots & \text{for } n_\mu>N,\\ \varphi_{1i_1}\otimes\cdots\otimes\varphi_{n_\mu i_{n_\mu}}\otimes\varphi_{n_\mu+1,0}^\vartheta\otimes\cdots\otimes\varphi_{N,0}^\vartheta\otimes\psi_{N+1,0}^F\otimes\cdots & \text{for } n_\mu\leqq N\end{cases}$$

where $\varphi_{\alpha 0}^\vartheta$, $\psi_{\alpha 0}^F$ is a component of respresentative of ϑ and F. That is to say $P_{\vartheta N}$ exchanges component vectors $\varphi_{\alpha\mu}$ such that $\varphi_{\alpha\mu}=\varphi_{\alpha 0}^\vartheta$ for $\alpha>N$ by vectors $\psi_{\alpha 0}^F$.

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3) I_α^H, and I_α are sets of indices which correspond to the Hilbert base and algebraic base respectively.

4) Haag's notation [7].

5) Here suffix F of $\varphi_{\alpha n}^F$ means free particle and $f_\alpha\bar{\in}F$.

For a vector $\Phi^{\vartheta} \in \Pi^{\vartheta} \frown \Pi' \otimes \mathfrak{H}_{\alpha}$, $\Phi^{\vartheta} = \sum_{\mu=1}^{l} c_{\mu} \Phi_{\mu}^{\vartheta}$ we define $P_{\vartheta N} \Phi^{\vartheta} = \sum_{\mu=1}^{l} c_{\mu} P_{\vartheta N} \Phi_{\mu}^{\vartheta}$.

Definition 2. P_N is a mapping from $\Pi' \otimes \mathfrak{H}_{\alpha}$ into $F \frown \Pi' \otimes \mathfrak{H}_{\alpha}$ defined by the following $P_N \Phi = P_N(\Sigma_{\vartheta} c_{\vartheta} \Phi^{\vartheta}) = \Sigma_{\vartheta} c_{\vartheta} P_{\vartheta N} \Phi^{\vartheta} = \Sigma_{\vartheta} c_{\vartheta} P_{\vartheta N}(\Sigma_{\mu} c_{\mu} \Phi_{\mu}^{\vartheta}) = \Sigma_{\vartheta} \Sigma_{\mu} c_{\vartheta} c_{\mu} P_{\vartheta N} \Phi_{\mu}^{\vartheta}$.

(5) For Π^{ϑ} whose representative is $\Pi \otimes \varphi_{\alpha 0}^{\vartheta}$, we can select especially the space $[\Pi' \otimes B_{\alpha}^{\vartheta}]$ such that for any α, $\varphi_{\alpha 0}^{\vartheta}$ and $\psi_{\alpha 0}^{F}$ is contained in $B_{\alpha} = \{\tilde{\varphi}_{\alpha\nu} | \nu\}$. We denote $[\widehat{\Pi} \otimes B_{\alpha}^{\vartheta}]$ or simply $[\widehat{\Pi}^{\vartheta}]$ such special $[\Pi \otimes B_{\alpha}^{\vartheta}]$. Let a representative Φ^{c} of Π^{c} belong to $[\widehat{\Pi}^{\vartheta}]$, then there exists a base of Π^{c} which makes a subset of $[\widehat{\Pi}^{\vartheta}]$. Using such a base for Π^{c} we can define cut-off operators P_N for a vector of $[\widehat{\Pi}^{\vartheta}]$ quite similarly in (4), and denote it by $\widehat{P}_N$.

3. **Direct method.** The problem related to the inequivalence of representation is solved by R. Haag and D. Castler [8] introducing a new topology. We show here that the problem is treated similarly using cut-off operator. To introduce the topology by which $\lim_{n\to\infty} P_n \Phi = \Phi$ for any state Φ of the universal Hilbert space, the neighborhood $U_N(\Phi)$ must contain Φ and $P_n \Phi$ for any n, $n \geqq N$. We can see also using $P_{FN} = 1$ that $U_N(\Phi) = \{\Phi, P_n \Phi | n \geqq N\}$ satisfies the postulates of neighborhood system of topological space $\Pi' \otimes \mathfrak{H}_{\alpha}$. This topology τ_f has following properties:

(1) τ_f is a finest one such that for any Φ, $P_n \Phi$ converges to Φ.

(2) The restriction of τ_f to F is a discrete topology.

(3) τ_f is not compatible with the linear operation i.e., $P_n \Phi + P_{n'}(\Phi) \to \Phi + \Phi$ does not follow from $P_n \Phi \to \Phi$ and $P_n \Psi \to \Psi$. (The topology used by Haag and Castler is similarly not compatible with the linear operations since $\phi_{\psi_1+\psi_2}(A) = \langle \psi_1 + \psi_2, A(\psi_1 + \psi_2) \rangle = \phi_{\psi_1}(A) + \phi_{\psi_2}(A) + 2R(\psi_2, A\psi_1)$.)

(4) T_1-separation axiom holds in this space.

The neighborhood of 0 of the locally convex topology which is compatible to linear operation must be of a form

$$U(0) = [(\Phi - P_{n'}\Phi), \lambda\Psi | n' \geqq N(\Phi), |\lambda| < \lambda_0(\Psi), \Phi, \Psi \in \Pi' \otimes \mathfrak{H}]^{c}\,{}^{6)}$$

where $N(\Phi)$ is a positive integer valued function of Φ and $\lambda_0(\Psi)$ is a positive valued function of Ψ, and with a few loss of generality for our purpose we may assume that

$$U_{N(\Phi),\varepsilon}(0) = [V_{\varepsilon} \frown F, \{\Phi - P_{n'}\Phi | n' > N(\Phi)\}]^{c}.$$

Now we restrict $N(\Phi)$ simply by $N(\Phi) \equiv N$, i.e.,

$$U_{N,\varepsilon}(0) = [U_{n \geqq N, \Phi \in E}\{\Phi - P_n \Phi\}]^{c} + V_{\varepsilon} \frown F^{7)}$$

and introduce topology τ_l. Similarly replacing P_N in $U_{N,\varepsilon}$ by $\widehat{P}_N$,[8]

6) $[\ ]^{c}$ means convex hull.

7) E means $\Pi' \otimes \mathfrak{H}_{\alpha}$ for P_N, and $[\widehat{\Pi} \otimes B_{\alpha}^{F}]$ for $\widehat{P}_N$.

8) $U_{N,\varepsilon}(\Phi_0)$ in [2], (p. 26) must be replaced by this neighborhood.

$\hat{\tau}_l$ is defined. τ_l and $\hat{\tau}_l$ are well defind by virtue of the following

Theorem 1.[9] *The cut-off operator* P_N *defines a locally convex topology* τ_l[10] *in* $\Pi'\otimes\mathfrak{H}_\alpha$ *such that the closure of* $F\frown\Pi'\otimes\mathfrak{H}_\alpha$ *includes* $\Pi'\otimes\mathfrak{H}_\alpha$. *The cut-off operator* $\hat{P}_N$ *defines a locally convex metrizable topology* $\hat{\tau}_l$ *in* $[\hat{\Pi}\otimes B^F]$ *such that the closure of* $F\frown[\hat{\Pi}\otimes B^F]$ *includes* $[\hat{\Pi}\otimes B^F]$.

Proof. The theorem is easily proved except separation axiom for $\hat{\tau}_l$. So we prove $\bigcap_{N,\varepsilon} U_{N,\varepsilon}=\{0\}$. Assume that there exists a vector x such that $x\in\bigcap_{N,\varepsilon} U_{N,\varepsilon}$, $x\neq 0$, $x\in[\hat{\Pi}]$.[11] Let $x=x^F+x^G$ where $x^F\in F$, $x^G\in[\hat{\Pi}]\ominus F\frown[\hat{\Pi}]$,[12] and let $x^G=\sum_{j=1}^s c_j\Phi_{\sigma_j}$ $(c_j\neq 0)$, $x^F=\sum_{k=1}^p d_k\Phi_{\omega_k}$ be expressions by the normalized base of $[\hat{\Pi}]$ which are given above.

Now there exist a number $\varepsilon>0$ such that $\varepsilon<\underset{1\leqq j\leqq s, 1\leqq k\leqq p}{\mathrm{Min}}(|c_j|,|d_k|)$ and an integer N such that for any $m_1,\cdots,m_{s+p}>N$ a system $P_{m_1}(\Phi_{\sigma_1}),\cdots,P_{m_{s+p}}(\Phi_{\omega_p})$ is linearly independent. Since $x\in U_{N,\varepsilon}$ for these N,ε, there exist vectors $\Psi_1,\cdots,\Psi_l,\eta$ and integers $n_1,\cdots,n_l>N$ such that $x=\sum_{i=1}^l(\Psi_i-P_{n_i}(\Psi_i))+\eta$ and $\|\eta\|<\varepsilon$.

Let $\Psi_\rho=\Psi_\rho^G+\Psi_\rho^F$ $(\rho=1,\cdots,l)$ and let $\Psi_\rho^G=\sum_{i=1}^s\lambda_{\rho i}\Phi_{\sigma_i}+\sum_{i=1}^t\mu_{\rho j}\Phi_{\tau_j}$ then $\sum_{\rho=1}^l\mu_{\rho j}=0$ $(j=1,\cdots,t)$ and $\sum_{\rho=1}^l\lambda_{\rho i}=c_i$ $(i=1,\cdots,s)$. Using $\hat{P}_{Fn}=1$ we obtain $x^F=\eta-\sum_{i=1}^s\{\sum_{\rho=1}^l\lambda_{\rho i}P_{n_\rho}\Phi_{\sigma_i}\}-\sum_{j=1}^t\{\sum_{\rho=1}^l\mu_{\rho j}P_{n_\rho}\Phi_{\tau_j}\}$. Since system $\{x^F, P_{n_j}\Phi_{\sigma_i}|i=1,\cdots,s\}$ are linearly independent and $\|\eta\|<\varepsilon$, there must exist Φ_{τ_i} such that $P_{n_\rho}(\Phi_{\tau_i})=k\cdot\Phi_{\omega_1}$ for a n_ρ. Without loss of generality we assume that $\Phi_{\tau_1},\cdots,\Phi_{\tau_m}$ have these properties, and $\Phi_{\tau_{m+1}},\cdots,\Phi_{\tau_l}$ have not such n_ρ $(\rho=1,\cdots,l)$. We write this fact explicitly by $\Phi_{\tau_{i(\omega_1)}}(1\leq i\leq m)$. Now we see easily for $1\leq i\leq m$, $P_{n_\nu}(\Phi_{\tau i})\neq k_j\Phi_{\omega_j}$ $(j\neq 1)$ and $\neq k_h\Phi_{\sigma_h}$ $(1\leq h\leq s)$, and hence a system $\{P_n\Phi_{\tau_{\lambda(\omega_1)}}, P_{n'}\Phi_{\tau_{\mu(\omega_j)}}(j\neq 1), P_{n''}\Phi_{\tau_{\nu(\sigma_i)}}\}$ for $n,n',n''>N$ is linearly independent. Hence we obtain $d_1\Phi_{\omega_1}=\eta'-\sum_{j=1}^m\{\sum_{\rho=1}^l\mu_{\rho j}P_{n_\rho}\Phi_{\tau j}\}$ where $\|\eta'\|\leqq\|\eta\|<\varepsilon$. The above selection of base $\hat{\Pi}$ ensures that for any family of base $\{\Phi_i|i=1,\cdots,m\}$ of $\hat{\Pi}$, $\{\hat{P}_n\Phi_i|i\}$ is a part of the base of $F\frown\hat{\Pi}\otimes^F B_\alpha$ except overlapping. Hence we see that if $\sum_{\rho=1}^l\mu_{\rho j}=0$ $(j=1,\cdots,m)$ then $\sum_{j=1}^m\sum_{\rho=1}^l(\mu_{\rho j}P_{n_\rho})\Phi_{\tau j}=(\sum_{j=1}^m\sum_{\rho=1}^l\mu_{\rho j})\Phi_{\omega_1}=0$. So we have $\varepsilon<|d_1|=\|d_1\Phi_{\omega_1}\|=\|\eta'\|<\varepsilon$, that is contradiction.

4. Dual space method. Since the dual of the space of state

9) As Example 1 shows $\hat{\Pi}\otimes B_\alpha^F$ contains $\mathfrak{h}$ in [2], (p. 26) and Theorem 1 is an extended results of [2].

10) We can not assure that τ_l satisfies T_2-separation axiom on the whole universal space.

11) $\hat{\Pi}$ means $\hat{\Pi}\otimes B^F$. $[\hat{\Pi}]$ means $[\hat{\Pi}\otimes B^F]$.

12) $[\hat{\Pi}]\ominus F\frown[\hat{\Pi}]$ means supplement, i.e. $[\hat{\Pi}]=([\hat{\Pi}]\frown F)\oplus([\hat{\Pi}]\ominus[\hat{\Pi}]\frown F)$ where $\oplus$ means direct sum.

vectors has fundamental physical meaning in quantum theory, it may be better to construct the space which preserves the value of inner product between state as much as possible. So we postulate that the inner products in F should be preserved though the orthogonality between F and $[\hat{\Pi}]\ominus F_{\frown}[\hat{\Pi}]$ must be removed.

From the construction of $[\hat{\Pi}]$, we can see that in the space F there is an algebraic base which contains all the maps of $P_n(\hat{\Pi})$ $(n=1, 2, 3, \cdots)$. We introduce elements of dual space defining the value on this base. For an element Φ of $F_{\frown}[\hat{\Pi}]$ we define an element $\hat{\Phi}$ of a base of a dual space by $\langle\hat{\Phi}, \Psi\rangle=1$ for $\Psi=\Phi$, and $\langle\hat{\Phi}, \Psi\rangle=0$ for $\Psi=\Phi$ where $\Psi\in[\hat{\Pi}]$. For an element Φ of $[\hat{\Pi}]\ominus F_{\frown}[\hat{\Pi}]$ we define an element $\hat{\Phi}$ of a dual space by the following

$$\langle\hat{\Phi}, \Psi\rangle=\begin{cases}1 & \text{if there exists } n \text{ such that } \Psi=P_n(\Phi) \text{ or } \Psi=\Phi.\\ 0 & \text{otherwise.}\end{cases}$$

Using S. Kasahara's theorem [4] [5], this assignment ensures the linear set $[\{\hat{\Phi}\}]$[13] makes a dual space $\hat{G}$ such that $\sigma([\hat{\Pi}], \hat{G})$ is a locally convex Hausdorff topology. Now in this space we can assure that for any element $\Phi\in[\hat{\Pi}]$, $\lim_{n\to\infty}\hat{P}_n\Phi=\Phi$ in the sense of $\sigma([\hat{\Pi}], \hat{G})$.

To prove this fact we have only to show that if $\Phi, \psi\in\hat{\Pi}$, $\Phi\neq\psi$ then there exists an integer N such that for any integer $n>N$, $\hat{P}_n\Phi\neq\hat{P}_n\Psi$. For $\Phi=\sum_{i=1}^{p}\lambda_i\Phi_{h_i}$, $\Psi=\sum_{j=1}^{q}\mu_j\Phi_{k_j}$ we make a base $(\Phi_{l_1}\cdots\Phi_{l_s})$ from a system $(\Phi_{h_1}\cdots\Phi_{h_p}\Phi_{k_1}\cdots\Phi_{k_q})$. Then there exists N such that $\hat{P}_N\Phi_{l_i}$ $(i=1, \cdots, s)$ make linearly independent family. Since there exist Φ_{l_i} such that $\lambda_{h_\rho}\neq\mu_{k_\nu}$ where λ_{h_ρ} and μ_{k_ν} are coefficients of Φ_{l_i} in a expression of Φ, Ψ respectively, $\hat{P}_n\Phi\neq\hat{P}_n\Psi$ for the component $\hat{P}_n\Phi_{l_i}$. Thus we obtain the following

Theorem 2. *$\hat{P}_n$ introduces locally convex Hausdorff topology τ such that for any $\Phi\in[\hat{\Pi}]$, $\hat{P}_n(\Phi)\in F$, $\lim\hat{P}_n(\Phi)=\Phi$ in the sense of $\sigma([\hat{\Pi}], \hat{G})$, and $\hat{G}$ contains $F_{\frown}[\hat{\Pi}]$ preserving ordinary orthogonality between elements of base as (Hilbert space).*

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13) [ ] means linear hull.

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