154. *Locally Convex Metrizable Topologies which Make a Given Vector Subspace Dense*

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1. The current theory of quantum field encounters with a difficulty arized from the fact that the orthogonality of the specific vector subspaces of the space of physical states denies the expansion of certain physical states.[1] It might be possible, however, to avoid this difficulty by changing the topology of the space of physical states. In the problem of this possibility, it is fundamental to show the existence of a topology having the property that a given vector subspace is dense in the whole space; moreover it will be desirable that the topology is locally convex and metrizable.[2]

The purpose of this paper is to give a condition under which the existence of such a topology is ensured.

The author is indebted to Professor T. Ishihara for suggesting this problem.

2. We shall concern exclusively with complex vector spaces. But every result obtained in what follows remains valid for real vector spaces.

Let E be a vector space, and let M be a vector subspace of E. A vector subspace N of E is called an *algebraic supplement* of M in E, if $M+N=E$ and $M\frown N=\{0\}$. A subset A of E is said to be *linearly independent* if a (finite) linear combination $\sum_{i=1}^{n}\lambda_i x_i$, where $x_i\in A$ for each i and $x_i\neq x_j$ for $i\neq j$, is 0 only when each λ_i is zero. By a *base* of E, we mean always a Hamel base of E, that is, a maximal linearly independent subset of E. We denote by $\dim(E)$ the dimension of E, i.e. the cardinal number of a base of E, and by $\operatorname{codim}(M)$ the codimension of M in E.

A pair (E, E') of vector spaces E and E' is called a *dual system* if a bilinear functional $\langle\cdot,\cdot\rangle$ on the product space $E\times E'$ is assigned. A dual system (E, E') is said to be *separated* if it satisfies the following two conditions:

1° If $\langle x, x'\rangle=0$ for all $x'\in E'$, then $x=0$.

2° If $\langle x, x'\rangle=0$ for all $x\in E$, then $x'=0$.

A dual system (E, E') which satisfies the condition 1° (resp. 2°) is

1) A brief survey of this circumstance can be found in T. Ishihara [1].

2) Cf. T. Ishihara [1].

said to be *separated in* E (resp. *in* E').

Let E be a topological vector space. We denote by E' the dual of E, and by E^* the algebraic dual of E.

3. We need the following

LEMMA. *Let E be an infinite dimensional metrizable topological vector space. Then the codimension of E' in E^* is not smaller than* $\aleph \cdot \dim(E)$.[3]

Proof. We denote by I the set of all positive integers, and by φ a one-to-one mapping of I onto $I \times I$. Let $\{V_i; i \in I\}$ be a fundamental system of neighborhoods of 0 in E. For every $i \in I$, we put $U_i = \frac{1}{n} V_m$, where $(n, m) = \varphi(i)$. Then $\{U_i; i \in I\}$ is also a fundamental system of neighborhoods of 0 in E. Let B be a base of E. Since the cardinal number of B is infinite, there exists a one-to-one mapping ψ of $B \times I$ onto B. Evidently we can assume, without loss of generality, that $\psi(x, n) \in U_n$ for each $(x, n) \in B \times I$. Now, for each $(y, \alpha) \in B \times (0, 1)$,[4] we define a function $y^{(\alpha)}$ on B by putting $y^{(\alpha)}(x) = \alpha \cdot C_y(x) + 1$ for all $x \in B$, where C_y is the characteristic function of the set $\{y\}$ consisting of single element y; and for each function $y^{(\alpha)}$, we define a linear functional $\bar{y}^{(\alpha)}$ on E by setting

$$\langle \psi(x, n), \bar{y}^{(\alpha)} \rangle = n^{y^{(\alpha)}(x)} \qquad \text{for all } (x, n) \in B \times I.$$

Consider a linear combination $y' = \sum_{i=1}^{k} \lambda_i \bar{y}_i^{(\alpha_i)}$, where $\bar{y}_i^{(\alpha_i)} \neq \bar{y}_j^{(\alpha_j)}$ for $i \neq j$. Since $i \neq j$ and $y_i = y_j$ imply $\alpha_i \neq \alpha_j$, it follows that, for each positive integer j, $1 \leqq j \leqq k$,

$$\langle \psi(y_j, n), y' \rangle = \sum_{i=1}^{k} \lambda_i \langle \psi(y_j, n), \bar{y}_i^{(\alpha_i)} \rangle$$

contains one and only one term of powder $\alpha_j + 1$. Therefore if $\lambda_j \neq 0$, there exists an $n_0 \in I$ such that

$$(*) \qquad |\langle \psi(y_j, n), y' \rangle| > 1 \qquad \text{for all } n \geqq n_0.$$

Consequently the set A' of all linear functional $\bar{y}^{(\alpha)}$, $(y, \alpha) \in B \times (0, 1)$, is linearly independent. In addition, because the mapping φ is one-to-one, we can find, for every $(l, m) \in I \times I$, a positive integer $i \geqq l$ such that $n = \varphi^{-1}(i, m) > n_0$; and hence we have $\psi(y_j, n) \in U_n = \frac{1}{i} V_m \subseteq \frac{1}{l} V_m$. Thus the inequality (*) shows that y' is not continuous if $\lambda_j \neq 0$. Therefore the vector subspace of E^* spanned by the set A' is contained in an algebraic supplement of E' in E^*. Now since the mapping $(y, \alpha) \to \bar{y}^{(\alpha)}$ of $B \times (0, 1)$ onto A' is one-to-one, the sets $B \times (0, 1)$ and A' have the same cardinal number. This completes the proof of the Lemma.

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3) We denote by $\aleph$ the cardinal number of the continuum.

4) The symbol $(0, 1)$ denotes the set of all real number x such that $0 < x < 1$.

The following corollary is an immediate consequence of the Lemma.

COROLLARY. *Let E be a vector space. Then the Mackey topology $\tau(E, E^*)$ is metrizable if and only if E is finite dimensional.*

4. We shall prove the following main theorem.

THEOREM 1. *Let M be an infinite dimensional vector subspace of a vector space E. If the codimension of M does not exceed $\aleph$, then for every locally convex metrizable topology τ_0 on M, and for every algebraic supplement N of M in E, there exists a locally convex metrizable topology τ on E satisfying the following conditions:*

(1) *M is dense in E.*

(2) *N is closed.*

(3) *The induced topology of τ on M is finer than τ_0.*

Proof. For each element x' of the dual M' of M for the topology τ_0, we define a linear functional $\bar{x}'$ on E by letting

$$\langle x, \bar{x}'\rangle = \begin{cases} \langle x, x'\rangle & \text{for } x \in M, \\ 0 & \text{for } x \in N. \end{cases}$$

By the Lemma, we can find a linearly independent countable (infinite) subset $\{y_i^*\}$ of an algebraic supplement of M' in M^*. Suppose that the dimension of N is infinite. Then since $\dim(N) \leqq \aleph = 2^{\aleph_0}$, by Lemma 4 of [2], there exists a vector subspace $N' \subseteq N^*$ of dimension $\leqq \aleph_0$ such that the dual system (N, N') is separated. Let $\{z_i'\}$ be a base of N; we define for each i, a linear functional $\bar{y}_i'$ by letting

$$\langle x, \bar{y}_i'\rangle = \begin{cases} \langle x, y_i^*\rangle & \text{for } x \in M, \\ \langle x, z_i'\rangle & \text{for } x \in N. \end{cases}$$

Let $\{V_i\}$ be a fundamental system of convex neighborhoods of 0 in M for the topology τ_0; then, for each i, the polar V_i° of V_i in M' is a $\sigma(M', M)$-compact convex circled set. We shall show that the set A_i' of all linear functionals $\bar{x}'$ on E, where $x' \in V_i^\circ$, is $\sigma(E^*, E)$-compact, convex and circled. To see the compactness, it will suffice to show that the mapping $x' \to \bar{x}'$ of M' into E^* is continuous for the topologies $\sigma(M', M)$ and $\sigma(E^*, E)$. Now each element $x \in E$ can be written uniquely in the form: $x = p(x) + y$, $p(x) \in M$, $y \in N$; hence for every finite subset $\{x_1, \cdots, x_n\}$ of E, if x' belongs to the polar of the set $\{p(x_1), \cdots, p(x_n)\}$ in M', then $\bar{x}'$ belongs to the polar of the set $\{x_1, \cdots, x_n\}$ in E^*. Thus the mapping $x' \to \bar{x}'$ is continuous, and so each A_i' is $\sigma(E^*, E)$-compact. It is clear that A_i' is convex and circled. Let us denote by E' the vector subspace of E^* spanned by the set $(\bigcup_{i=1}^{\infty} A_i') \cup \{\bar{y}_i';\ i = 1, 2, \cdots\}$. Then the dual system (E, E') is separated. In fact, let x be a non-zero element of E; if $p(x) \neq 0$, then we have $\langle x, \bar{x}'\rangle = \langle p(x), x'\rangle \neq 0$ for some $x' \in M'$; if $p(x) = 0$, then since $x \in N$, we have

$\langle x, z'_i \rangle \neq 0$ for some z'_i, and so $\langle x, y'_i \rangle \neq 0$. Now the topology τ of uniform convergence on each member of the family of sets $\{A'_i; i=1, 2, \cdots\} \cup \{\{\bar{y}'_i\}; i=1, 2, \cdots\}$ has the required property. Clearly, the topology τ is locally convex and metrizable which satisfies the condition (3). In addition, the dual of E for this topology is E', as can be easily seen. It remains to establish (1) and (2). If x does not belong to N, then since $p(x) \neq 0$, we can find an $x' \in M'$ such that $\langle p(x), x' \rangle \neq 0$, and hence we have $\langle x, \bar{x}' \rangle \neq 0$. Because $\bar{x}'$ vanishes on N, this shows that N is closed. Let x'_0 be an element of E' which vanishes on M. We can write $x'_0 = \bar{x}' + \sum_{i=1}^{n} \lambda_i \bar{y}'_i$, where $x' \in M'$. Hence we have for every $x \in M$, $0 = \langle x, x'_0 \rangle = \langle x, \bar{x}' + \sum_{i=1}^{n} \lambda_i \bar{y}'_i \rangle = \langle x, x' + \sum_{i=1}^{n} \lambda_i y^*_i \rangle$, and so $x' + \sum_{i=1}^{n} \lambda_i y^*_i$ vanishes on M. It follows that $x'=0$ and $\lambda_1 = \cdots = \lambda_n = 0$, since the set $\{x', y^*_1, \cdots, y^*_n\}$ is linearly independent. Consequently we have $M^\circ \frown E' = \{0\}$, which shows that M is dense in E for the topology τ.

Now, suppose that the dimension of the vector subspace N is finite n. Let $\{z_1, \cdots, z_n\}$ be a base of N; we define, for each $i=1, \cdots, n$, a linear functional z'_i on N by setting

$$\langle z_j, z'_i \rangle = \delta_{ij}, \qquad j=1, \cdots, n,$$

where δ_{ij} is the Kronecker's delta. Then the proof proceeds quite similarly by taking $\{y^*_i; i=1, \cdots, n\}$ and $\{z'_i; i=1, \cdots, n\}$ instead of $\{y^*_i\}$ and $\{z'_i\}$ respectively, and we omit the detail.

Since every vector space can be introduced a norm topology, we have as a corollary of Theorem 1 the following

THEOREM 2. *Let E be a vector space, and let M be an infinite dimensional vector subspace of E with codimension $\leqq \aleph$. Then for every algebraic supplement N of M in E, there exists a locally convex metrizable topology on E for which the conditions* (1) *and* (2) *of Theorem* 1 *hold.*

By Theorem 3 of [2], these theorems yield the following

COROLLARY. *Let E be a vector space, and let M, N be two vector subspaces of E. Suppose that the dimension of M is infinite, and the codimension of M in the vector space $M+N$ does not exceed $\aleph$. Then*

1° *For every locally convex metrizable topology τ_0 on M, there exists a locally convex metrizable topology τ on E satisfying the following conditions*:

(1) *The closure of M contains N.*

(2) *The induced topology of τ on M is finer than τ_0.*

2° *There exists a locally convex metrizable topology on E such that the closure of M contains N.*

(This article is dedicated to Professor K. Kunugi is celebration of his 60th birthday.)

References

[1] T. Ishihara: On some topologies in the universal Hilbert space. Proc. Japan Acad., **40**, 23–28 (1964).
[2] S. Kasahara: On closures of vector subspaces. I. Proc. Japan Acad., **40**, 723–727 (1964).